A Gentle Introduction to the Universal Algorithmic Agent AIXI

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Keywords

Abstract

Decision theory formally solves the problem of rational agents in uncertain worlds if the true environmental prior probability distribution is known. Solomonoff's theory of universal induction formally solves the problem of sequence prediction for unknown prior distribution. We combine both ideas and get a parameter-free theory of universal Artificial Intelligence. We give strong arguments that the resulting AIXI model is the most intelligent unbiased agent possible. We outline for a number of problem classes, including sequence prediction, strategic games, function minimization, reinforcement and supervised learning, how the AIXI model can formally solve them. The major drawback of the AIXI model is that it is uncomputable. To overcome this problem, we construct a modified algorithm AIXItl, which is still effectively more intelligent than any other time t and space l bounded agent. The computation time of AIXItl is of the order $t \cdot 2^l$. Other discussed topics are formal definitions of intelligence order relations, the horizon problem and relations of the AIXI theory to other AI approaches.

Contents

1   Introduction

This chapter gives an introduction to a mathematical theory for intelligence. We present the AIXI model, a parameter-free optimal reinforcement learning agent embedded in an arbitrary unknown environment.

The science of Artificial Intelligence (AI) may be defined as the construction of intelligent systems and their analysis. A natural definition of *systems* is anything which has an input and an output stream. Intelligence is more complicated. It can have many faces like creativity, solving problems, pattern recognition, classification, learning, induction, deduction, building analogies, optimization, surviving in an environment, language processing, knowledge, and many more. A formal definition incorporating every aspect of intelligence, however, seems difficult. Most, if not all known facets of intelligence can be formulated as goal driven or, more precisely, as maximizing some utility function. It is, therefore, sufficient to study goal driven AI; e.g. the (biological) goal of animals and humans is to survive and spread. The goal of AI systems should be to be useful to humans. The problem is that, except for special cases, we know neither the utility function, nor the environment in which the agent will operate, in advance. The mathematical theory, coined AIXI, is supposed to solve these problems.

Assume the availability of unlimited computational resources. The first important observation is that this makes the AI problem not trivial. Playing chess optimally or solving NP-hard problems become trivial, but driving a car or surviving in nature don't. The reason being, that it is a challenge itself to well-define these problems, not to mention to present an algorithm. In other words: The AI problem has not yet been well-defined. One may view AIXI as a suggestion for such a mathematical definition of AI.

AIXI is a universal theory of sequential decision making akin to Solomonoff's celebrated universal theory of induction. Solomonoff derived an optimal way of predicting future data, given previous observations, provided the data is sampled from a computable

probability distribution. AIXI extends this approach to an optimal decision making agent embedded in an unknown environment. The *main idea* is to replace the unknown environmental distribution μ in the Bellman equations by a suitably generalized universal Solmononoff distribution ξ. The state space is the space of complete histories. AIXI is a universal theory without adjustable parameters, making no assumptions about the environment except that it is sampled from a computable distribution. From an algorithmic complexity perspective, the AIXI model generalizes optimal passive universal induction to the case of active agents. From a decision theoretic perspective, AIXI is a suggestion of a new (implicit) "learning" algorithm, which may overcome all (except computational) problems of previous reinforcement learning algorithms.

There are strong arguments that AIXI is the most intelligent unbiased agent possible. We outline for a number of problem classes, including sequence prediction, strategic games, function minimization, reinforcement and supervised learning, how the AIXI model can formally solve them. The major drawback of the AIXI model is that it is incomputable. To overcome this problem, we construct a modified algorithm AIXI*tl*, which is still effectively more intelligent than any other time t and space l bounded agent. The computation time of AIXI*tl* is of the order $t \cdot 2^l$. Other discussed topics are formal definitions of intelligence order relations, the horizon problem and relations of the AIXI theory to other AI approaches.

The article is meant to be a gentle introduction to and discussion of the AIXI model. For a mathematically rigorous treatment, many subtleties, and proofs see the references to the author's works in the annotated bibliography section at the end. This section also provides references to introductory textbooks and original publications on algorithmic information theory and sequential decision theory.

Chapter 2 presents the theory of sequential decisions in a very general form (called AIμ model) in which actions and observations may depend on arbitrary past events. We clarify the connection to the Bellman equations and discuss minor parameters including (the size of) the I/O spaces and the lifetime of the agent and their universal choice which we have in mind. Optimality of AIμ is obvious by construction.

Chapter 3: How and in which sense induction is possible at all has been subject to long philosophical controversies. Highlights are Epicurus' principle of multiple explanations, Occam's razor and Bayes' rule for conditional probabilities. Solomonoff elegantly unified all these aspects into one formal theory of inductive inference based on a universal probability distribution ξ, which is closely related to Kolmogorov complexity $K(x)$, the length of the shortest program computing x. Rapid convergence of ξ to to the unknown true environmental distribution μ and tight loss bounds for arbitrary bounded loss functions and finite alphabet can be shown. Pareto-optimality of ξ in the sense that there is no other predictor which performs better or equal in all environments and strictly better in at least one can also be shown. In view of these results it is fair to say that the problem of sequence prediction possesses a universally optimal solution.

Chapter 4: In the active case, reinforcement learning algorithms are usually used if μ is unknown. They can succeed if the state space is either small or has effectively been made small by generalization techniques. The algorithms work only in restricted

(e.g. Markovian) domains, have problems with optimally trading off exploration versus exploitation, have nonoptimal learning rate, are prone to diverge, or are otherwise ad hoc. The formal solution proposed here is to generalize Solomonoff's universal prior ξ to include action conditions and replace μ by ξ in the AIμ model, resulting in the AIξ≡AIXI model, which we claim to be universally optimal. We investigate what we can expect from a universally optimal agent and clarify the meanings of *universal*, *optimal*, etc. Other discussed topics are formal definitions of an intelligence order relation, the horizon problem, and Pareto-optimality of AIXI.

Chapter 5: We show how a number of AI problem classes fit into the general AIXI model. They include sequence prediction, strategic games, function minimization, and supervised learning. We first formulate each problem class in its natural way (for known μ) and then construct a formulation within the AIμ model and show their equivalence. We then consider the consequences of replacing μ by ξ. The main goal is to understand in which sense the problems are solved by AIXI.

Chapter 6: The major drawback of AIXI is that it is incomputable, or more precisely, only asymptotically computable, which makes an implementation impossible. To overcome this problem, we construct a modified model AIXItl, which is still superior to any other time t and space l bounded algorithm. The computation time of AIXItl is of the order $t \cdot 2^l$. The solution requires an implementation of first order logic, the definition of a universal Turing machine within it and a proof theory system.

Chapter 7: Finally we discuss and remark on some otherwise unmentioned topics of general interest. We remark on various topics, including concurrent actions and observations, the choice of the I/O spaces, treatment of encrypted information, and peculiarities of mortal embodies agents. We continue with an outlook on further research, including optimality, down-scaling, implementation, approximation, elegance, extra knowledge, and training of/for AIXI(tl). We also include some (personal) remarks on non-computable physics, the number of wisdom Ω, and consciousness.

An annotated bibliography and other references conclude this work.

2   Agents in Known Probabilistic Environments

The general framework for AI might be viewed as the design and study of intelligent agents [RN95]. An agent is a cybernetic system with some internal state, which acts with output y_k on some environment in cycle k, perceives some input x_k from the environment and updates its internal state. Then the next cycle follows. We split the input x_k into a regular part x'_k and a reward r_k, often called reinforcement feedback. From time to time the environment provides non-zero reward to the agent. The task of the agent is to maximize its utility, defined as the sum of future rewards. A probabilistic environment can be described by the conditional probability μ for the inputs $x_1...x_n$ to the agent under the condition that the agent outputs $y_1...y_n$. Most, if not all environments are of this type. We give formal expressions for the outputs of the agent, which maximize the total μ-expected reward sum, called value. This model is called the AIμ model. As every AI

problem can be brought into this form, the problem of maximizing utility is hence being formally solved, if μ is known. Furthermore, we study some special aspects of the AIμ model. We introduce factorizable probability distributions describing environments with independent episodes. They occur in several problem classes studied in Section 5 and are a special case of more general separable probability distributions defined in Section 4.3. We also clarify the connection to the Bellman equations of sequential decision theory and discuss similarities and differences. We discuss minor parameters of our model, including (the size of) the input and output spaces $\mathcal{X}$ and $\mathcal{Y}$ and the lifetime of the agent, and their universal choice, which we have in mind. There is nothing remarkable in this section, it is the essence of sequential decision theory [NM44, Bel57, BT96, SB98], presented in a new form. Notation and formulas needed in later sections are simply developed. There are two major remaining problems. The problem of the unknown true probability distribution μ, which is solved in Section 4, and computational aspects, which are addressed in Section 6.

2.1   The Cybernetic Agent Model

A good way to start thinking about intelligent systems is to consider more generally cybernetic systems, in AI usually called agents. This avoids having to struggle with the meaning of intelligence from the very beginning. A cybernetic system is a control circuit with input y and output x and an internal state. From an external input and the internal state the agent calculates deterministically or stochastically an output. This output (action) modifies the environment and leads to a new input (perception). This continues ad infinitum or for a finite number of cycles.

Definition 1 (The Agent Model) *An agent is a system which interacts with an environment in cycles $k = 1,2,3,....$. In cycle k the action (output) $y_k \in \mathcal{Y}$ of the agent is determined by a policy p which depends on the I/O-history $y_1 x_1 ... y_{k-1} x_{k-1}$. The environment reacts to this action and leads to a new perception (input) $x_k \in \mathcal{X}$ determined by a deterministic function q or probability distribution μ which depends on the history $y_1 x_1 ... y_{k-1} x_{k-1} y_k$. Then the next cycle $k+1$ starts.*

As explained in the last section, we need some reward assignment to the cybernetic system. The input x is divided into two parts, the standard input x' and some reward input r. If input and output are represented by strings, a deterministic cybernetic system can be modeled by a Turing machine p. p is called the policy of the agent, which determines the (re)action to a perception. If the environment is also computable it might be modeled by a Turing machine q as well. The interaction of the agent with the environment can be illustrated as follows:
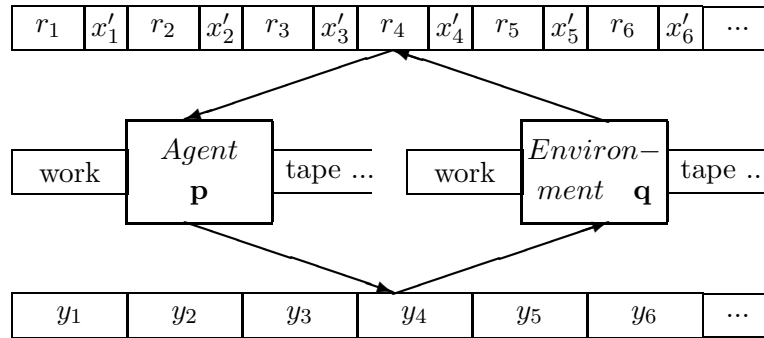
p as well as q have unidirectional input and output tapes and bidirectional working tapes. What entangles the agent with the environment, is the fact that the upper tape serves as input tape for p, as well as output tape for q, and that the lower tape serves as output tape for p as well as input tape for q. Further, the reading head must always be left of the writing head, i.e. the symbols must first be written, before they are read. p and q have their own mutually inaccessible working tapes containing their own 'secrets'. The heads move in the following way. In the kth cycle p writes y_k, q reads y_k, q writes $x_k \equiv x'_k r_k$, p reads $x_k \equiv x'_k r_k$, followed by the $(k+1)^{th}$ cycle and so on. The whole process starts with the first cycle, all heads on tape start and working tapes being empty. We want to call Turing machines behaving in this way, *chronological Turing machines*. Before continuing, some notations on strings and probability distributions are appropriate.

2.2 Strings

We will denote strings over the alphabet $\mathcal{X}$ by $s = x_1 x_2 ... x_n$, with $x_k \in \mathcal{X}$, where $\mathcal{X}$ is alternatively interpreted as a non-empty subset of $I\!N$ or itself as a prefix free set of binary strings. $l(s) = l(x_1) + ... + l(x_n)$ is the length of s. Analogous definitions hold for $y_k \in \mathcal{Y}$. We call x_k the kth input word and y_k the kth output word (rather than letter). The string $s = y_1 x_1 ... y_n x_n$ represents the input/output in chronological order. Due to the prefix property of the x_k and y_k, s can be uniquely separated into its words. The words appearing in strings are always in chronological order. We further introduce the following abbreviations: ϵ is the empty string, $x_{n:m} := x_n x_{n+1} ... x_{m-1} x_m$ for $n \le m$ and ϵ for $n > m$. $x_{<n} := x_1 ... x_{n-1}$. Analogously for y. Further, $yx_n := y_n x_n$, $yx_{n:m} := y_n x_n ... y_m x_m$, and so on.

2.3 AI model for Known Deterministic Environment

Let us define for the chronological Turing machine p a partial function also named $p : \mathcal{X}^* \to \mathcal{Y}^*$ with $y_{1:k} = p(x_{<k})$ where $y_{1:k}$ is the output of Turing machine p on input $x_{<k}$ in cycle k, i.e. where p has read up to x_{k-1} but no further[1] In an analogous way, we define $q : \mathcal{Y}^* \to \mathcal{X}^*$ with $x_{1:k} = q(y_{1:k})$. Conversely, for every partial recursive chronological function we can define a corresponding chronological Turing machine. Each (agent,environment) pair (p,q)

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[1]Note that a possible additional dependence of p on $y_{<k}$ as mentioned in Definition 1 can be eliminated by recursive substitution; see below. Similarly for q.

produces a unique I/O sequence $\omega^{pq} := y_1^{pq} x_1^{pq} y_2^{pq} x_2^{pq}....$ When we look at the definitions of p and q we see a nice symmetry between the cybernetic system and the environment. Until now, not much intelligence is in our agent. Now the credit assignment comes into the game and removes the symmetry somewhat. We split the input $x_k \in \mathcal{X} := \mathcal{R} \times \mathcal{X}'$ into a regular part $x_k' \in \mathcal{X}'$ and a reward $r_k \in \mathcal{R} \subset I\!\!R$. We define $x_k \equiv x_k' r_k$ and $r_k \equiv r(x_k)$. The goal of the agent should be to maximize received rewards. This is called reinforcement learning. The reason for the asymmetry is, that eventually we (humans) will be the environment with which the agent will communicate and *we* want to dictate what is good and what is wrong, not the other way round. This one way learning, the agent learns from the environment, and not conversely, neither prevents the agent from becoming more intelligent than the environment, nor does it prevent the environment learning from the agent because the environment can itself interpret the outputs y_k as a regular and a reward part. The environment is just not forced to learn, whereas the agent is. In cases where we restrict the reward to two values $r \in \mathcal{R} = I\!\!B$, $r = 1$ is interpreted as a positive feedback, called *good* or *correct* and $r = 0$ a negative feedback, called *bad* or *error*. Further, let us restrict for a while the lifetime (number of cycles) m of the agent to a large, but finite value. Let $V_{km}^{pq} := \sum_{i=k}^m r(x_i)$ be the future total reward (called future utility), the agent p receives from the environment q in the cycles k to m. It is now natural to call the agent p^*, which maximizes V_{1m} (called total utility), the *best* one[2].

$$p^* := \arg\max_p V_{1m}^{pq} \quad \Rightarrow \quad V_{km}^{p^*q} \geq V_{km}^{pq} \quad \forall p : y_{<k}^{pq} = y_{<k}^{p^*q} \tag{1}$$

For $k = 1$ the condition on p is nil. For $k > 1$ it states that p shall be consistent with p^* in the sense that they have the same history. If $\mathcal{X}$, $\mathcal{Y}$ and m are finite, the number of different behaviors of the agent, i.e. the search space is finite. Therefore, because we have assumed that q is known, p^* can effectively be determined (by pre-analyzing all behaviors). The main reason for restricting to finite m was not to ensure computability of p^* but that the limit $m \to \infty$ might not exist. The ease with which we defined and computed the optimal policy p^* is not remarkable. Just the (unrealistic) assumption of a completely known deterministic environment q has trivialized everything.

2.4   AI Model for Known Prior Probability

Let us now weaken our assumptions by replacing the environment q with a probability distribution $\mu(q)$ over chronological functions. μ might be interpreted in two ways. Either the environment itself behaves stochastically defined by μ or the true environment is deterministic, but we only have subjective (probabilistic) information, of which environment being the true environment. Combinations of both cases are also possible. We assume here that μ is known and describes the true stochastic behavior of the environment. The case of unknown μ with the agent having some beliefs about the environment lies at the heart of the AIξ model described in Section 4.

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[2]$\arg\max_p V(p)$ is the p which maximizes $V(\cdot)$. If there is more than one maximum we might choose the lexicographically smallest one for definiteness.

The *best* or *most intelligent* agent is now the one which maximizes the *expected* utility (called value function) $V_\mu^p \equiv V_{1m}^{p\mu} := \sum_q \mu(q) V_{1m}^{pq}$. This defines the AI$\mu$ model.

Definition 2 (The AIμ model) *The AIμ model is the agent with policy p^μ which maximizes the μ-expected total reward $r_1 + ... + r_m$, i.e. $p^* \equiv p^\mu := \mathrm{argmax}_p V_\mu^p$. $V_\mu^* := V_\mu^{p^\mu}$.*

We need the concept of a *value function* in a slightly more general form.

Definition 3 (The μ/true/generating value function) *The agent's perception x consists of a regular input $x' \in \mathcal{X}'$ and a reward $r \in \mathcal{R} \subset \mathrm{I\!R}$. In cycle k the value $V_{km}^{p\mu}(y x_{<k})$ is defined as the μ-expectation of the future reward sum $r_k + ... + r_m$ with actions generated by policy p, and fixed history $y x_{<k}$. We say that $V_{km}^{p\mu}(y x_{<k})$ is the (future) value of policy p in environment μ given history $y x_{<k}$, or shorter, the μ or true or generating value of p given $y x_{<k}$. $V_\mu^p := V_{1m}^{p\mu}$ is the (total) value of p.*

We now give a more formal definition for $V_{km}^{p\mu}$. Let us assume we are in cycle k with history $\ddot{y}\ddot{x}_1 ... \ddot{y}\ddot{x}_{k-1}$ and ask for the *best* output y_k. Further, let $\dot{Q}_k := \{q : q(\dot{y}_{<k}) = \dot{x}_{<k}\}$ be the set of all environments producing the above history. We say that $q \in \dot{Q}_k$ is *consistent* with history $\ddot{y}\ddot{x}_{<k}$. The expected reward for the next $m-k+1$ cycles (given the above history) is called the value of policy p and is given by a conditional probability:

$$V_{km}^{p\mu}(\ddot{y}\ddot{x}_{<k}) \; := \; \frac{\sum_{q \in \dot{Q}_k} \mu(q) V_{km}^{pq}}{\sum_{q \in \dot{Q}_k} \mu(q)}. \tag{2}$$

Policy p and environment μ do not determine history $\ddot{y}\ddot{x}_{<k}$ unlike the deterministic case because the history is no longer deterministically determined by p and q, but depends on p and μ *and* on the outcome of a stochastic process. Every new cycle adds new information $(\dot{x}_i)$ to the agent. This is indicated by the dots over the symbols. In cycle k we have to maximize the expected future rewards, taking into account the information in the history $\ddot{y}\ddot{x}_{<k}$. This information is not already present in p and q/μ at the agent's start unlike in the deterministic case.

Furthermore, we want to generalize the finite lifetime m to a dynamical (computable) farsightedness $h_k \equiv m_k - k + 1 \geq 1$, called horizon. For $m_k = m$ we have our original finite lifetime, for $h_k = h$ the agent maximizes in every cycle the next h expected rewards. A discussion of the choices for m_k is delayed to Section 4.5.

The next h_k rewards are maximized by

$$p_k^* \; := \; \arg \max_{p \in \dot{P}_k} V_{km_k}^{p\mu}(\ddot{y}\ddot{x}_{<k}),$$

where $\dot{P}_k := \{p : \exists y_k : p(\dot{x}_{<k}) = \dot{y}_{<k} y_k\}$ is the set of systems consistent with the current history. p_k^* depends on k and is used only in step k to determine $\dot{y}_k$ by $p_k^*(\dot{x}_{<k}|\dot{y}_{<k}) = \dot{y}_{<k}\dot{y}_k$. After writing $\dot{y}_k$ the environment replies with $\dot{x}_k$ with (conditional) probability $\mu(\dot{Q}_{k+1})/\mu(\dot{Q}_k)$. This probabilistic outcome provides new information to the agent. The cycle $k+1$ starts with determining $\dot{y}_{k+1}$ from p_{k+1}^* (which differs from p_k^* as $\dot{x}_k$ is now fixed) and so on.

Note that p_k^* implicitly depends also on $\dot{y}_{<k}$ because $\dot{P}_k$ and $\dot{Q}_k$ do so. But recursively inserting p_{k-1}^* and so on, we can define

$$p^*(\dot{x}_{<k}) \;:=\; p_k^*(\dot{x}_{<k}|p_{k-1}^*(\dot{x}_{<k-1}|...p_1^*)) \tag{3}$$

It is a chronological function and computable if $\mathcal{X}$, $\mathcal{Y}$ and m_k are finite and μ is computable. For constant m one can show that the policy (3) coincides with the AIμ model (2). This also proves

$$V_{km}^{*\mu}(yx_{<k}) \geq V_{km}^{p\mu}(yx_{<k}) \quad \forall p \text{ consistent with } yx_{<k} \tag{4}$$

similarly to (1). For $k=1$ this is obvious. We also call (3) AIμ model. For deterministic[3] μ this model reduces to the deterministic case discussed in the last subsection.

It is important to maximize the sum of future rewards and not, for instance, to be greedy and only maximize the next reward, as is done e.g. in sequence prediction. For example, let the environment be a sequence of chess games and each cycle corresponds to one move. Only at the end of each game a positive reward $r=1$ is given to the agent if it won the game (and made no illegal move). For the agent, maximizing all future rewards means trying to win as many games in as short as possible time (and avoiding illegal moves). The same performance is reached, if we choose h_k much larger than the typical game lengths. Maximization of only the next reward would be a very bad chess playing agent. Even if we would make our reward r finer, e.g. by evaluating the number of chessmen, the agent would play very bad chess for $h_k=1$, indeed.

The AIμ model still depends on μ and m_k. m_k is addressed in Section 4.5. To get our final universal AI model the idea is to replace μ by the universal probability ξ, defined later. This is motivated by the fact that ξ converges to μ in a certain sense for any μ. With ξ instead of μ our model no longer depends on any parameters, so it is truly universal. It remains to show that it behaves intelligently. But let us continue step by step. In the following we develop an alternative but equivalent formulation of the AIμ model. Whereas the functional form presented above is more suitable for theoretical considerations, especially for the development of a time bounded version in Section 6, the iterative/recursive formulation of the next subsections will be more appropriate for the explicit calculations in most of the other sections.

2.5   Probability Distributions

We use Greek letters for probability distributions and underline their arguments to indicate that they are probability arguments. Let $\rho_n(\underline{x}_1...\underline{x}_n)$ be the probability that an (infinite) string starts with $x_1...x_n$. We drop the index on ρ if it is clear from its arguments:

$$\sum_{x_n\in\mathcal{X}} \rho(\underline{x}_{1:n}) \equiv \sum_{x_n} \rho_n(\underline{x}_{1:n}) = \rho_{n-1}(\underline{x}_{<n}) \equiv \rho(\underline{x}_{<n}), \quad \rho(\epsilon) \equiv \rho_0(\epsilon) = 1. \tag{5}$$

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[3]We call a probability distribution deterministic if it is 1 for exactly one argument and 0 for all others.

We also need conditional probabilities derived from Bayes' rule. We prefer a notation which preserves the chronological order of the words, in contrast to the standard notation $\rho(\cdot|\cdot)$ which flips it. We extend the definition of ρ to the conditional case with the following convention for its arguments: An underlined argument $\underline{x}_k$ is a probability variable and other non-underlined arguments x_k represent conditions. With this convention, Bayes' rule has the form $\rho(x_{<n}\underline{x}_n) = \rho(\underline{x}_{1:n})/\rho(\underline{x}_{<n})$. The equation states that the probability that a string $x_1...x_{n-1}$ is followed by x_n is equal to the probability of $x_1...x_n*$ divided by the probability of $x_1...x_{n-1}*$. We use $x*$ as an abbreviation for 'strings starting with x'.

The introduced notation is also suitable for defining the conditional probability $\rho(y_1\underline{x}_1...y_n\underline{x}_n)$ that the environment reacts with $x_1...x_n$ under the condition that the output of the agent is $y_1...y_n$. The environment is chronological, i.e. input x_i depends on $yx_{<i}y_i$ only. In the probabilistic case this means that $\rho(\underline{yx}_{<k}y_k):=\sum_{x_k}\rho(\underline{yx}_{1:k})$ is independent of y_k, hence a tailing y_k in the arguments of ρ can be dropped. Probability distributions with this property will be called *chronological*. The y are always conditions, i.e. never underlined, whereas additional conditioning for the x can be obtained with Bayes' rule

$$\rho(yx_{<n}\underline{yx}_n) = \rho(\underline{yx}_{1:n})/\rho(\underline{yx}_{<n}) \quad \text{and} \tag{6}$$

$$\rho(\underline{yx}_{1:n}) = \rho(\underline{yx}_1)\cdot\rho(yx_1\underline{yx}_2)\cdot ... \cdot\rho(yx_{<n}\underline{yx}_n) \tag{7}$$

The second equation is the first equation applied n times.

2.6 Explicit Form of the AIμ Model

Let us define the AIμ model p^* in a different way: Let $\mu(yx_{<k}\underline{yx}_k)$ be the true probability of input x_k in cycle k, given the history $yx_{<k}y_k$. $\mu(\underline{yx}_{1:k})$ is the true chronological prior probability that the environment reacts with $x_{1:k}$ if provided with actions $y_{1:k}$ from the agent. We assume the cybernetic model depicted on page 6 to be valid. Next we define the value $V^{*\mu}_{k+1,m}(yx_{1:k})$ to be the μ-expected reward sum $r_{k+1}+...+r_m$ in cycles $k+1$ to m with outputs y_i generated by agent p^*, which maximizes the expected reward sum, and responses x_i from the environment, drawn according to μ. Adding $r(x_k)\equiv r_k$ we get the reward including cycle k. The probability of x_k, given $yx_{<k}y_k$, is given by the conditional probability $\mu(yx_{<k}\underline{yx}_k)$. So the expected reward sum in cycles k to m given $yx_{<k}y_k$ is

$$V^{*\mu}_{km}(yx_{<k}y_k) := \sum_{x_k}[r(x_k) + V^{*\mu}_{k+1,m}(yx_{1:k})]\cdot\mu(yx_{<k}\underline{yx}_k) \tag{8}$$

Now we ask about how p^* chooses y_k. It should choose y_k as to maximize the future rewards. So the expected reward in cycles k to m given $yx_{<k}$ and y_k chosen by p^* is $V^{*\mu}_{km}(yx_{<k}):=\max_{y_k}V^{*\mu}_{km}(yx_{<k}y_k)$ (see Figure 4).

Together with the induction start

$$V^{*\mu}_{m+1,m}(yx_{1:m}) := 0 \tag{9}$$

$V^{*\mu}_{km}$ is completely defined. We might summarize one cycle into the formula

$$V^{*\mu}_{km}(yx_{<k}) = \max_{y_k}\sum_{x_k}[r(x_k) + V^{*\mu}_{k+1,m}(yx_{1:k})]\cdot\mu(yx_{<k}\underline{yx}_k) \tag{10}$$

$$V_{km}^{*\mu}(y\!x_{<k}) = \max_{y_k} V_{km}^{*\mu}(y\!x_{<k}y_k)$$

$$\underbrace{max}$$

$$y_k=0 \qquad y_k=1$$

action y_k with max value.

$$V_{km}^{*\mu}(y\!x_{<k}y_k) = \sum_{x_k}[r_k + V_{k+1,m}^{*\mu}(y\!x_{1:k})]\mu(y\!x_{<k}y\underline{x}_k)$$

$$\mathbf{E} \qquad\qquad \mathbf{E}$$

$$x_k'=0 \quad x_k'=1 \quad x_k'=0 \quad x_k'=1$$
$$r_k=... \quad r_k=... \quad r_k=... \quad r_k=...$$

μ expected reward r_k and observation x_k'.

$$V_{k+1,m}^{*\mu}(y\!x_{1:k}) = \max_{y_{k+1}} V_{k+1,m}^{*\mu}(y\!x_{1:k}y_{k+1})$$

$$\underbrace{max}\ y_{k+1}\quad \underbrace{max}\ y_{k+1}\quad \underbrace{max}\ y_{k+1}\quad \underbrace{max}$$

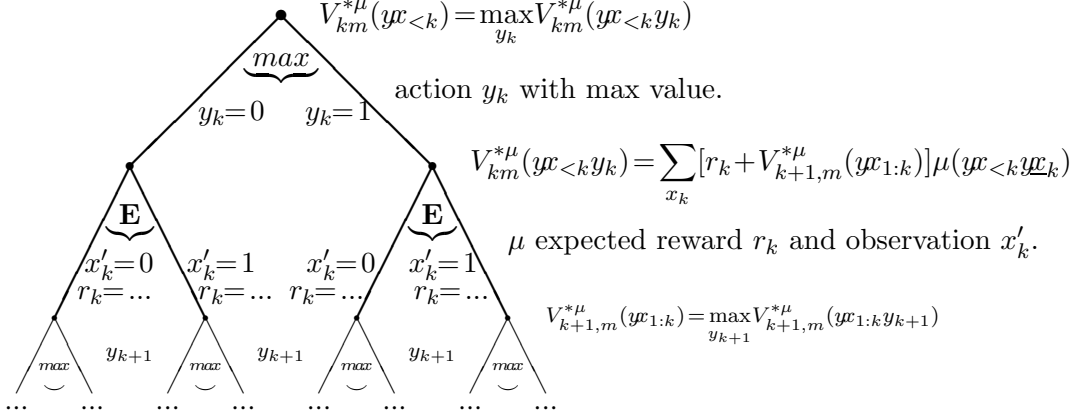$$...\quad ...\quad\quad ...\quad ...\quad\quad ...\quad ...\quad\quad ...\quad ...$$

Figure 4 (Expectimax Tree/Algorithm for $\mathcal{X}'=\mathcal{Y}=\mathbb{B}$)

We introduce a dynamical (computable) farsightedness $h_k \equiv m_k - k + 1 \geq 1$, called horizon. For $m_k = m$, where m is the lifetime of the agent, we achieve optimal behavior, for limited farsightedness $h_k = h$ ($m = m_k = h + k - 1$), the agent maximizes in every cycle the next h expected rewards. A discussion of the choices for m_k is delayed to Section 4.5. If m_k is our horizon function of p^* and $\dot{y}\!\dot{x}_{<k}$ is the actual history in cycle k, the output $\dot{y}_k$ of the agent is explicitly given by

$$\dot{y}_k \;=\; \arg\max_{y_k} V_{km_k}^{*\mu}(\dot{y}\!\dot{x}_{<k}y_k) \tag{11}$$

which in turn defines the policy p^*. Then the environment responds $\dot{x}_k$ with probability $\mu(\dot{y}\!\dot{x}_{<k}\dot{y}\underline{\dot{x}}_k)$. Then cycle $k+1$ starts. We might unfold the recursion (10) further and give $\dot{y}_k$ non-recursively as

$$\dot{y}_k \;\equiv\; \dot{y}_k^\mu \;:=\; \arg\max_{y_k} \sum_{x_k} \max_{y_{k+1}} \sum_{x_{k+1}} \;...\; \max_{y_{m_k}} \sum_{x_{m_k}} (r(x_k) + ... + r(x_{m_k}))\cdot\mu(\dot{y}\!\dot{x}_{<k}y\underline{x}_{k:m_k}) \tag{12}$$

This has a direct interpretation: the probability of inputs $x_{k:m_k}$ in cycle k when the agent outputs $y_{k:m_k}$ with actual history $\dot{y}\!\dot{x}_{<k}$ is $\mu(\dot{y}\!\dot{x}_{<k}y\underline{x}_{k:m_k})$. The future reward in this case is $r(x_k) + ... + r(x_{m_k})$. The best expected reward is obtained by averaging over the x_i ($\sum_{x_i}$) and maximizing over the y_i. This has to be done in chronological order to correctly incorporate the dependency of x_i and y_i on the history. This is essentially the expectimax algorithm/tree [Mic66, RN95]. The AIμ model is *optimal* in the sense that no other policy leads to higher expected reward. The value for a general policy p can be written in the form

$$V_{km}^{p\mu}(y\!x_{<k}) \;:=\; \sum_{x_{1:m}} (r_k + ... + r_m)\mu(y\!x_{<k}y\underline{x}_{k:m})_{|y_{1:m}=p(x_{<m})} \tag{13}$$

As is clear from their interpretations, the iterative environmental probability μ relates to the functional form in the following way:

$$\mu(y\underline{x}_{1:k}) \;=\; \sum_{q:q(y_{1:k})=x_{1:k}} \mu(q) \tag{14}$$

With this identification one can show [Hut00] the following:

Theorem 5 (Equivalence of functional an iterative AI model) *The actions of the functional AI model (3) coincide with the actions of the recursive/iterative AI model (10-12) with environments identified by (14).*

2.7 Factorizable Environments

Up to now we have made no restrictions on the form of the prior probability μ apart from being a chronological probability distribution. On the other hand, we will see that, in order to prove rigorous reward bounds, the prior probability must satisfy some separability condition to be defined later. Here we introduce a very strong form of separability, when μ factorizes into products.

Assume that the cycles are grouped into independent episodes $r=1,2,3,...$, where each episode r consists of the cycles $k=n_r+1,...,n_{r+1}$ for some $0=n_0<n_1<...<n_s=n$:

$$\mu(y\underline{x}_{1:n}) = \prod_{r=0}^{s-1} \mu_r(y\underline{x}_{n_r+1:n_{r+1}}) \tag{15}$$

(In the simplest case, when all episodes have the same length l then $n_r=r{\cdot}l$). $\dot{y}_k$ depends on μ_r and x and y of episode r only, with r such that $n_r<k\leq n_{r+1}$. One can show that

$$\dot{y}_k = \arg\max_{y_k} V_{km_k}^{*\mu}(\dot{y}\dot{x}_{<k}y_k) = \arg\max_{y_k} V_{kt}^{*\mu}(\dot{y}\dot{x}_{<k}y_k) \tag{16}$$

with $t:=\min\{m_k,n_{r+1}\}$. The different episodes are completely independent in the sense that the inputs x_k of different episodes are statistically independent and depend only on y_k of the same episode. The outputs y_k depend on the x and y of the corresponding episode r only, and are independent of the actual I/O of the other episodes.

Note, that $\dot{y}_k$ is also independent of the choice of m_k, as long as m_k is sufficiently large. If all episodes have a length of at most l, i.e. $n_{r+1}-n_r \leq l$ and if we choose the horizon h_k to be at least l, then $m_k \geq k+l-1 \geq n_r+l \geq n_{r+1}$ and hence $t=n_{r+1}$ independent of m_k. This means that for factorizable μ there is no problem in taking the limit $m_k \to \infty$. Maybe this limit can also be performed in the more general case of a sufficiently separable μ. The (problem of the) choice of m_k will be discussed in more detail later.

Although factorizable μ are too restrictive to cover all AI problems, it often occurs in practice in the form of repeated problem solving, and hence, is worth being studied. For example, if the agent has to play games like chess repeatedly, or has to minimize different functions, the different games/functions might be completely independent, i.e. the environmental probability factorizes, where each factor corresponds to a game/function minimization. For details, see the appropriate sections on strategic games and function minimization.

Further, for factorizable μ it is probably easier to derive suitable reward bounds for the universal AIξ model defined in the next section, than for the separable cases which

will be introduced later. This could be a first step toward a definition and proof for the general case of separable problems. One goal of this paragraph was to show, that the notion of a factorizable μ could be the first step toward a definition and analysis of the general case of separable μ.

2.8   Constants and Limits

We have in mind a universal agent with complex interactions that is as least as intelligent and complex as a human being. One might think of a agent whose input y_k comes from a digital video camera, the output x_k is some image to a monitor[4], only for the valuation we might restrict to the most primitive binary one, i.e. $r_k \in I\!B$. So we think of the following constant sizes:

$$
\begin{array}{ccccccccc}
1 & \ll & \langle l(y_k x_k) \rangle & \ll & k & \leq & m & \ll & |\mathcal{Y} \times \mathcal{X}| \\
1 & \ll & 2^{16} & \ll & 2^{24} & \leq & 2^{32} & \ll & 2^{65536}
\end{array}
$$

The first two limits say that the actual number k of inputs/outputs should be reasonably large, compared to the typical size $\langle l \rangle$ of the input/output words, which itself should be rather sizeable. The last limit expresses the fact that the total lifetime m (number of I/O cycles) of the agent is far too small to allow every possible input to occur, or to try every possible output, or to make use of identically repeated inputs or outputs. We do not expect any useful outputs for $k \lesssim \langle l \rangle$. More interesting than the lengths of the inputs is the complexity $K(x_1...x_k)$ of all inputs until now, to be defined later. The environment is usually not "perfect". The agent could either interact with a non-perfect human or tackle a non-deterministic world (due to quantum mechanics or chaos)[5]. In either case, the sequence contains some noise, leading to $K \propto \langle l \rangle \cdot k$. The complexity of the probability distribution of the input sequence is something different. We assume that this noisy world operates according to some simple computable rules. $K(\mu_k) \ll \langle l \rangle \cdot k$, i.e. the rules of the world can be highly compressed. We may allow environments in which new aspects appear for $k \to \infty$ causing a non-bounded $K(\mu_k)$.

   In the following we never use these limits, except when explicitly stated. In some simpler models and examples the size of the constants will even violate these limits (e.g. $l(x_k) = l(y_k) = 1$), but it is the limits above that the reader should bear in mind. We are only interested in theorems which do not degenerate under the above limits. In order to avoid cumbersome convergence and existence considerations we make the following assumptions throughout this work.

Assumption 6 (Finiteness) *We assume that*

- *the input/perception space $\mathcal{X}$ is finite,*
- *the output/action space $\mathcal{Y}$ is finite,*
- *the rewards are non-negative and bounded, i.e. $r_k \in \mathcal{R} \subseteq [0, r_{max}]$,*

---

[4]Humans can only simulate a screen as output device by drawing pictures.

[5]Whether there exist truly stochastic processes at all is a difficult question. At least the quantum indeterminacy comes very close to it.

- *the horizon m is finite.*

Finite $\mathcal{X}$ and bounded $\mathcal{R}$ (each separately) ensure existence of μ-expectations, but are sometimes needed together, finite $\mathcal{Y}$ ensures that $\mathrm{argmax}_{y_k \in \mathcal{Y}}[...]$ exists, i.e. that maxima are attained, finite m avoids various technical and philosophical problems (Section 4.5), positive rewards are needed for the time bounded AIξ^{tl} model (Section 6). Many theorems can be generalized by relaxing some or all of the above finiteness assumptions.

2.9 Sequential Decision Theory

One can relate (10) to the Bellman equations [Bel57] of sequential decision theory by identifying complete histories $yx_{<k}$ with states, $\mu(yx_{<k}y\underline{x}_k)$ with the state transition matrix, V_μ^* with the value function, and y_k with the action in cycle k [BT96, RN95]. Due to the use of complete histories as state space, the AIμ model neither assumes stationarity, nor the Markov property, nor complete accessibility of the environment. Every state occurs at most once in the lifetime of the system. For this and other reasons the explicit formulation (12) is much more useful here than to enforce a pseudo-recursive Bellman equation form.

As we have in mind a universal system with complex interactions, the action and perception spaces $\mathcal{Y}$ and $\mathcal{X}$ are huge (e.g. video images), and every action or perception itself occurs usually only once in the lifespan m of the agent. As there is no (obvious) universal similarity relation on the state space, an effective reduction of its size is impossible, but there is no principle problem in determining y_k from (12) as long as μ is known and computable and $\mathcal{X}$, $\mathcal{Y}$ and m are finite.

Things dramatically change if μ is unknown. Reinforcement learning algorithms [KLM96, SB98, BT96] are commonly used in this case to learn the unknown μ. They succeed if the state space is either small or has effectively been made small by generalization or function approximation techniques. In any case, the solutions are either *ad hoc*, work in restricted domains only, have serious problems with state space exploration versus exploitation, are prone to diverge, or have non-optimal learning rate. There is no universal and optimal solution to this problem so far. The central theme of this article is to present a new model and argue that it formally solves all these problems in an optimal way. The true probability distribution μ will not be learned directly, but will be replaced by some universal prior ξ, which converges to μ.

3 Universal Sequence Prediction

This section deals with the question of how to make predictions in unknown environments. Following a brief description of important philosophical attitudes regarding inductive reasoning and inference, we describe more accurately what we mean by induction, and motivate why we can focus on sequence prediction tasks. The most important concept is Occam's razor (simplicity) principle. Indeed, one can show that the best way to make predictions is based on the shortest ($\hat{=}$ simplest) description of the data sequence seen

so far. The most general effective descriptions can be obtained with the help of general recursive functions, or equivalently by using programs on Turing machines, especially on the universal Turing machine. The length of the shortest program describing the data is called the Kolmogorov complexity of the data. Probability theory is needed to deal with uncertainty. The environment may be a stochastic process (e.g. gambling houses or quantum physics) which can be described by "objective" probabilities. But also uncertain knowledge about the environment, which leads to beliefs about it, can be modeled by "subjective" probabilities. The old question left open by subjectivists of how to choose the a priori probabilities is solved by Solomonoff's universal prior, which is closely related to Kolmogorov complexity. Solomonoff's major result is that the universal (subjective) prior converges to the true (objective) environment(al probability) μ. The only assumption on μ is that μ (which needs not be known!) is computable. The problem of the unknown environment μ is hence solved for all problems of inductive type, like sequence prediction and classification.

3.1   Introduction

An important and highly non-trivial aspect of intelligence is inductive inference. Simply speaking, induction is the process of predicting the future from the past or, more precisely, it is the process of finding rules in (past) data and using these rules to guess future data. Weather or stock-market forecasting, or continuing number series in an IQ test, are non-trivial examples. Making good predictions plays a central role in natural and artificial intelligence in general, and in machine learning in particular. All induction problems can be phrased as sequence prediction tasks. This is, for instance, obvious for time series prediction, but also includes classification tasks. Having observed data x_t at times $t < n$, the task is to predict the nth symbol x_n from sequence $x_1...x_{n-1}$. This *prequential approach* [Daw84] skips over the intermediate step of learning a model based on observed data $x_1...x_{n-1}$ and then using this model to predict x_n. The prequential approach avoids problems of model consistency, how to separate noise from useful data, and many other issues. The goal is to make "good" predictions, where the prediction quality is usually measured by a loss function, which shall be minimized. The key concept to well-define and solve induction problems is *Occam's razor* (simplicity) principle, which says that "*Entities should not be multiplied beyond necessity,*" which may be interpreted as to keep the simplest theory consistent with the observations $x_1...x_{n-1}$ and to use this theory to predict x_n. Before we can present Solomonoff's formal solution, we have to quantify Occam's razor in terms of Kolmogorov complexity, and introduce the notion of subjective/objective probabilities.

3.2   Algorithmic Information Theory

Intuitively a string is simple if it can be described in a few words, like "the string of one million ones", and is complex if there is no such short description, like for a random string whose shortest description is specifying it bit-by-bit. We can restrict the discussion to

binary strings, since for other (non-stringy mathematical) objects we may assume some default coding as binary strings. Furthermore, we are only interested in effective descriptions, and hence restrict decoders to be Turing machines. Let us choose some universal (so called prefix) *Turing machine U* with unidirectional binary input and output tapes and a bidirectional working tape. We can then define the (conditional) *prefix Kolmogorov complexity* [Cha75, Gác74, Kol65, Lev74] of a binary string x as the length l of the shortest program p, for which U outputs the binary string x (given y)

Definition 7 (Kolmogorov complexity) *Let U be a universal prefix Turing machine U. The (conditional) prefix Kolmogorov complexity is defined as the shortest program p, for which U outputs x (given y):*

$$K(x) := \min_p\{l(p) : U(p) = x\}, \quad K(x|y) := \min_p\{l(p) : U(y, p) = x\}$$

Simple strings like 000...0 can be generated by short programs, and, hence have low Kolmogorov complexity, but irregular (e.g. random) strings are their own shortest description, and hence have high Kolmogorov complexity. An important property of K is that it is nearly independent of the choice of U. Furthermore it shares many properties with Shannon's entropy (information measure) S, but K is superior to S in many respects. To be brief, K is an excellent universal complexity measure, suitable for quantifying Occam's razor. There is (only) one severe disadvantage: K is not finitely computable. The major algorithmic property of K is that it is (only) co-enumerable.

For general (non-string) objects one can specify some default coding and define $K(object) := K(\langle object\rangle)$, especially for numbers and pairs, e.g. we abbreviate $K(x,y) := K(\langle x,y\rangle)$. The most important information theoretic properties of K are listed below, where we abbreviate $f(x) \leq g(x) + O(1)$ by $f(x) \stackrel{+}{\leq} g(x)$. We also later abbreviate $f(x) = O(g(x))$ by $f(x) \stackrel{\times}{\leq} g(x)$.

Theorem 8 (Information properties of Kolmogorov complexity)

i) $K(x) \stackrel{+}{\leq} l(x) + 2\log l(x), \qquad K(n) \stackrel{+}{\leq} \log n + 2\log\log n$

ii) $\sum_x 2^{-K(x)} \leq 1, \quad K(x) \geq l(x)$ *for "most"* $x, \quad K(n) \to \infty$ *for* $n \to \infty$.

iii) $K(x|y) \stackrel{+}{\leq} K(x) \stackrel{+}{\leq} K(x,y)$

iv) $K(x,y) \stackrel{+}{\leq} K(x) + K(y), \qquad K(xy) \stackrel{+}{\leq} K(x) + K(y)$

v) $K(f(x)) \stackrel{+}{\leq} K(x) + c_f$ *if* $f : \mathbb{B}^* \to \mathbb{B}^*$ *is recursive/computable*

vi) $K(x|y, K(y)) + K(y) \stackrel{+}{=} K(x,y) \stackrel{+}{=} K(y,x) \stackrel{+}{=} K(y|x, K(x)) + K(x)$

vii) $K(x) \stackrel{+}{\leq} -\log_2 P(x) + K(P)$ *if* $P : \mathbb{B}^* \to [0,1]$ *is recursive and* $\sum_x P(x) \leq 1$

All (in)equalities remain valid if K is (further) conditioned under some z, i.e. $K(...) \rightsquigarrow K(...|z)$ and $K(...|y) \rightsquigarrow K(...|y,z)$. Those stated are all valid within an additive constant of size $O(1)$, but there are others which are only valid to logarithmic accuracy. K has many properties in common with Shannon entropy as it should be, since both measure the information content of a string. (i) gives an upper bound on K. (ii) is Kraft's inequality which implies a lower bound on K valid for "most" n, where "most" means that there are only $o(N)$ exceptions for $n \in \{1,...,N\}$. Providing side information y can never increase code-length, requiring extra information y can never decrease code-length (iii). Coding x and y separately never helps (iv), and transforming x does not increase its information content (v). (v) also shows that switching from one coding scheme to another by means of a recursive bijection leaves K unchanged within additive $O(1)$ terms. The first non-trivial result is the symmetry of information (vi), which is the analogue of Bayes' rule. (vii) is at the hart of the MDL principle [Ris89], which approximates $K(x)$ by $-\log_2 P(x) + K(P)$.

3.3   Uncertainty & Probabilities

For the *objectivist* probabilities are real aspects of the world. The outcome of an observation or an experiment is not deterministic, but involves physical random processes. Kolmogorov's axioms of probability theory formalize the properties which probabilities should have. In the case of i.i.d. experiments the probabilities assigned to events can be interpreted as limiting frequencies (*frequentist* view), but applications are not limited to this case. Conditionalizing probabilities and Bayes' rule are the major tools in computing posterior probabilities from prior ones. For instance, given the initial binary sequence $x_1...x_{n-1}$, what is the probability of the next bit being 1? The probability of observing x_n at time n, given past observations $x_n...x_{n-1}$ can be computed with Bayes' rule[6] if the true generating distribution μ of the sequences $x_1 x_2 x_3...$ is known: $\mu(x_{<n}\underline{x}_n) = \mu(\underline{x}_{1:n})/\mu(\underline{x}_{<n})$ (see Sections 2.2 and 2.5). The problem, however, is that one often does not know the true distribution μ (e.g. in the cases of weather and stock-market forecasting).

The *subjectivist* uses probabilities to characterize an agent's degree of belief in (or plausibility of) something, rather than to characterize physical random processes. This is the most relevant interpretation of probabilities in AI. It is somewhat surprising that plausibilities can be shown to also respect Kolmogorov's axioms of probability and Bayes' rule by assuming only a few plausible qualitative rules they should follow [Cox46]. Hence, if the plausibility of $x_{1:n}$ is $\xi(\underline{x}_{1:n})$, the degree of belief in x_n given $x_{<n}$ is, again, given by Bayes' rule: $\xi(x_{<n}\underline{x}_n) = \xi(\underline{x}_{1:n})/\xi(\underline{x}_{<n})$.

Bayes' rule allows computing posterior probabilities/plausibilities from prior ones, but leaves open the question of how to determine the priors themselves. In statistical physics, the principle of indifference (symmetry principle) and the maximum entropy principle can often be exploited to determine prior probabilities, but only Occam's razor is general enough to assign prior probabilities in *every* situation, especially to cope with domains typical for AI.

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[6]Strictly speaking it is just the definition of conditional probabilities.

3.4   Algorithmic Probability & Universal Induction

Occam's razor (appropriately interpreted and in compromise with Epicurus' principle of indifference) tells us to assign high/low a priori plausibility to simple/complex strings x. Using K as the complexity measure any monotone decreasing function of K, e.g. $\xi(\underline{x}) = 2^{-K(x)}$ would satisfy this criterion. But ξ also has to satisfy the probability axioms, so we have to be a bit more careful. Solomonoff [Sol64, Sol78] defined the *universal prior* $\xi(\underline{x})$ as the probability that the output of a universal Turing machine U starts with x when provided with fair coin flips on the input tape. Formally, ξ can be defined as

$$\xi(\underline{x}) := \sum_{p\,:\,U(p)=x*} 2^{-l(p)} \geq 2^{-K(x)} \tag{17}$$

where the sum is over all (so called minimal) programs p for which U outputs a string starting with x. The inequality follows by dropping all terms in $\sum_p$ except for the shortest p computing x. Strictly speaking ξ is only a *semimeasure* since it is not normalized to 1, but this is acceptable/correctable. We derive the following bound:

$$\sum_{t=1}^{\infty}(1-\xi(x_{<t}\underline{x}_t))^2 \;\leq\; -\tfrac{1}{2}\sum_{t=1}^{\infty}\ln\xi(x_{<t}\underline{x}_t) \;=\; -\tfrac{1}{2}\ln\xi(\underline{x}_{1:\infty}) \;\leq\; \tfrac{1}{2}\ln 2\cdot K(x_{1:\infty})$$

In the first inequality we have used $(1-a)^2 \leq -\tfrac{1}{2}\ln a$ for $0 \leq a \leq 1$. In the equality we exchanged the sum with the logarithm and eliminated the resulting product by Bayes' rule (6). In the last inequality we used (17). If $x_{1:\infty}$ is a computable sequence, then $K(x_{1:\infty})$ is finite, which implies $\xi(x_{<t}\underline{x}_t) \to 1$ $(\sum_{t=1}^{\infty}(1-a_t)^2 < \infty \Rightarrow a_t \to 1)$. This means, that if the environment is a computable sequence (whichsoever, e.g. the digits π or e in binary representation), after having seen the first few digits, ξ correctly predicts the next digit with high probability, i.e. it recognizes the structure of the sequence.

   Assume now that the true sequence is drawn from the distribution μ, i.e. the true (objective) probability of $x_{1:n}$ is $\mu(\underline{x}_{1:n})$, but μ is unknown. How is the posterior (subjective) belief $\xi(x_{<n}\underline{x}_n) = \xi(\underline{x}_n)/\xi(\underline{x}_{<n})$ related to the true (objective) posterior probability $\mu(x_{<n}\underline{x}_n)$? Solomonoff's [Sol78] crucial result is that the posterior (subjective) beliefs converge to the true (objective) posterior probabilities, if the latter are computable. More precisely he showed that

$$\sum_{t=1}^{\infty}\sum_{x_{<t}}\mu(\underline{x}_{<t})\Big(\xi(x_{<t}\underline{0}) - \mu(x_{<t}\underline{0})\Big)^2 \;\overset{+}{\leq}\; \tfrac{1}{2}\ln 2\cdot K(\mu), \tag{18}$$

$K(\mu)$ is finite if μ is computable, but the infinite sum on the l.h.s. can only be finite if the difference $\xi(x_{<t}\underline{0}) - \mu(x_{<t}\underline{0})$ tends to zero for $t \to \infty$ with μ probability 1. This shows that using ξ as an estimate for μ may be a reasonable thing to do.

3.5   Loss Bounds & Pareto-Optimality

Most predictions are eventually used as a basis for some decision (or action), which itself leads to some reward or loss. Let $\ell_{x_t y_t} \in [0,1] \subset I\!\!R$ be the received loss when performing

prediction/decision/action $y_t \in \mathcal{Y}$ and $x_t \in \mathcal{X}$ is the tth symbol of the sequence. Let $y_t^\Lambda \in \mathcal{Y}$ be the prediction of a (causal) prediction scheme Λ. The true probability of the next symbol being x_t, given $x_{<t}$, is $\mu(x_{<t}\underline{x}_t)$. The expected loss when predicting y_t is $\mathbf{E}[\ell_{x_t y_t}]$. The total μ-expected loss suffered by the Λ scheme in the first n predictions is

$$L_n^\Lambda \;:=\; \sum_{t=1}^n \mathbf{E}[\ell_{x_t y_t^\Lambda}] \;=\; \sum_{t=1}^n \sum_{x_{1:t} \in \mathcal{X}^t} \mu(\underline{x}_{1:t}) \ell_{x_t y_t^\Lambda} \tag{19}$$

The goal is to minimize the expected loss. More generally, we define the Λ_ρ sequence prediction scheme (later also called SPρ) $y_t^{\Lambda_\rho} := \mathrm{argmin}_{y_t \in \mathcal{Y}} \sum_{x_t} \rho(x_{<t}\underline{x}_t) \ell_{x_t y_t}$ which minimizes the ρ-expected loss. If μ is known, Λ_μ is obviously the best prediction scheme in the sense of achieving minimal expected loss ($L_n^{\Lambda_\mu} \leq L_n^\Lambda$ for any Λ). We have proven the following loss bound for the universal Λ_ξ predictor [Hut01b, Hut01a]

$$0 \;\leq\; L_n^{\Lambda_\xi} - L_n^{\Lambda_\mu} \;\leq\; 2\ln 2 \cdot K(\mu) + 2\sqrt{L_n^{\Lambda_\mu} \ln 2 \cdot K(\mu)} \tag{20}$$

Together with $L_n \leq n$ this shows that $\frac{1}{n} L_n^{\Lambda_\xi} - \frac{1}{n} L_n^{\Lambda_\mu} = O(n^{-1/2})$, i.e. asymptotically Λ_ξ achieves the optimal average loss of Λ_μ with rapid convergence. Moreover $L_\infty^{\Lambda_\xi}$ is finite if $L_\infty^{\Lambda_\mu}$ is finite and $L_n^{\Lambda_\xi}/L_n^{\Lambda_\mu} \to 1$ if $L_\infty^{\Lambda_\mu}$ is not finite. Bound (20) also implies $L_n^\Lambda \geq L_n^{\Lambda_\xi} - 2\sqrt{L_n^{\Lambda_\xi} \ln 2 \cdot K(\mu)}$, which shows that *no* (causal) predictor Λ whatsoever achieves significantly less (expected) loss than Λ_ξ. In view of these results it is fair to say that, ignoring computational issues, the problem of sequence prediction has been solved in a universal way.

A different kind of optimality is *Pareto-optimality*. The universal prior ξ is Pareto-optimal in the sense that there is no other predictor which leads to equal or smaller loss in *all* environments. Any improvement achieved by some predictor Λ over Λ_ξ in some environments is balanced by a deterioration in other environments [Hut02b].

4   The Universal Algorithmic Agent AIXI

Active systems, like game playing (SG) and optimization (FM), cannot be reduced to induction systems. The *main idea of this work* is to generalize universal induction to the general agent model described in Section 2. For this, we generalize ξ to include conditions and replace μ by ξ in the rational agent model, resulting in the AIξ(=AIXI) model. In this way the problem that the true prior probability μ is usually unknown is solved. Convergence of $\xi \to \mu$ will be shown indicating that the AIξ model could behave optimally in any computable but unknown environment with reinforcement feedback.

The main focus of this section is to investigate what we can expect from a universally optimal agent and to clarify the meanings of *universal, optimal*, etc. Unfortunately bounds similar to the loss bound (19) in the SP case can hold for *no* active agent. This forces us to lower our expectation about universally optimal agents and to introduce other (weaker)

performance measures. Finally, we show that AIξ is Pareto-optimal in the sense that there is no other policy yielding higher or equal value in *all* environments and a strictly higher value in at least one.

4.1 The Universal AIξ Model

Definition of the AIξ model. We have developed enough formalism to suggest our universal AIξ model. All we have to do is to suitably generalize the universal semimeasure ξ from the last subsection and replace the true but unknown prior probability μ^{AI} in the AIμ model by this generalized ξ^{AI}. In what sense this AIξ model is universal will be discussed later.

In the functional formulation we define the universal probability ξ^{AI} of an environment q just as $2^{-l(q)}$

$$\xi(q) \;:=\; 2^{-l(q)}$$

The definition could not be easier[7]![8] Collecting the formulas of Section 2.4 and replacing $\mu(q)$ by $\xi(q)$ we get the definition of the AIξ agent in functional form. Given the history $\ddot{y}\ddot{x}_{<k}$ the policy p^ξ of the functional AIξ agent is given by

$$\dot{y}_k \;:=\; \arg\max_{y_k} \max_{p:p(\dot{x}_{<k})=\dot{y}_{<k}y_k} \sum_{q:q(\dot{y}_{<k})=\dot{x}_{<k}} 2^{-l(q)} \cdot V_{km_k}^{pq} \tag{21}$$

in cycle k, where $V_{km_k}^{pq}$ is the total reward of cycles k to m_k when agent p interacts with environment q. We have dropped the denominator $\sum_q \mu(q)$ from (2) as it is independent of the $p \in \dot{P}_k$ and a constant multiplicative factor does not change argmax_{y_k}.

For the iterative formulation the universal probability ξ can be obtained by inserting the functional $\xi(q)$ into (14)

$$\xi(y\underline{x}_{1:k}) \;=\; \sum_{q:q(y_{1:k})=x_{1:k}} 2^{-l(q)} \tag{22}$$

Replacing μ by ξ in (12) the iterative AIξ agent outputs

$$\dot{y}_k \;\equiv\; \dot{y}_k^\xi \;:=\; \arg\max_{y_k} \sum_{x_k} \max_{y_{k+1}} \sum_{x_{k+1}} \; ... \; \max_{y_{m_k}} \sum_{x_{m_k}} (r(x_k)+...+r(x_{m_k})) \cdot \xi(\ddot{y}\ddot{x}_{<k}y\underline{x}_{k:m_k}) \tag{23}$$

in cycle k given the history $\ddot{y}\ddot{x}_{<k}$.

The equivalence of the functional and iterative AI model proven in Section 2.6 is true for every chronological semimeasure ρ, especially for ξ, hence we can talk about *the* AIξ model in this respect. It (slightly) depends on the choice of the universal Turing machine. $l(\lceil q \rceil)$ is defined only up to an additive constant. The AIξ model also depends on the choice of $\mathcal{X} = \mathcal{R} \times \mathcal{X}'$ and $\mathcal{Y}$, but we do not expect any bias when the spaces are

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[7]It is not necessary to use $2^{-K(q)}$ or something similar as some reader may expect at this point. The reason is that for every program q there exists a functionally equivalent program q' with $K(q')=l(q')$.

[8]Here and later we identify objects with their coding relative to some fixed Turing machine U. For example, if q is a function $K(q):=K(\lceil q \rceil)$ with $\lceil q \rceil$ being a binary coding of q such that $U(\lceil q \rceil,y)=q(y)$. On the other hand, if q already is a binary string we define $q(y):=U(q,y)$.

chosen sufficiently simple, e.g. all strings of length 2^{16}. Choosing $I\!N$ as the word space would be optimal, but whether the maxima (suprema) exist in this case, has to be shown beforehand. The only non-trivial dependence is on the horizon function m_k which will be discussed later. So apart from m_k and unimportant details the AIξ agent is uniquely defined by (21) or (23). It doesn't depend on any assumption about the environment apart from being generated by some computable (but unknown!) probability distribution.

Convergence of ξ to μ. Similarly to (18) one can show that the μ-expected squared difference of μ and ξ is finite for computable μ. This, in turn, shows that $\xi(yx_{<k}y\underline{x}_k)$ converges to $\mu(yx_{<k}y\underline{x}_k)$ for $k \to \infty$ with μ probability 1. The line of reasoning is the same; the y are, again, pure spectators. This will change when we analyze loss/reward bounds analogous to (20).

If we take a finite product of $\xi's$ and use Bayes' rule, we see that also $\xi(yx_{<k}y\underline{x}_{k:k+r})$ converges to $\mu(yx_{<k}y\underline{x}_{k:k+r})$. More generally, in case of a bounded horizon h_k, it follows that[9]

$$\xi(yx_{<k}y\underline{x}_{k:m_k}) \overset{k \to \infty}{\longrightarrow} \mu(yx_{<k}y\underline{x}_{k:m_k}) \quad \text{if} \quad h_k \equiv m_k - k + 1 \leq h_{max} < \infty \qquad (24)$$

This makes us confident that the outputs $\dot{y}_k$ of the AIξ model (23) could converge to the outputs $\dot{y}_k$ from the AIμ model (12), at least for bounded horizon.

We want to call an AI model *universal*, if it is μ independent (unbiased, model-free) and is able to solve any solvable problem and learn any learnable task. Further, we call a universal model, *universally optimal*, if there is no program, which can solve or learn significantly faster (in terms of interaction cycles). As the AIξ model is parameter-free, ξ converges to μ (24), the AIμ model is itself optimal, and we expect no other model to converge faster to AIμ by analogy to SP (20):

Claim 9 (We expect AIXI to be universally optimal)

This is our main claim. In a sense, the intention of the remaining sections is to define this statement more rigorously and to give further support.

Intelligence order relation. We define the ξ-expected reward in cycles k to m of a policy p similar to (2) and (21). We extend the definition to programs $p \notin \dot{P}_k$ which are not consistent with the current history.

$$V_{km}^{p\xi}(\dot{y}\dot{x}_{<k}) := \frac{1}{\mathcal{N}} \sum_{q:q(\dot{y}_{<k})=\dot{x}_{<k}} 2^{-l(q)} \cdot V_{km}^{\tilde{p}q} \qquad (25)$$

The normalization $\mathcal{N}$ is again only necessary for interpreting V_{km} as the expected reward but otherwise unneeded. For consistent policies $p \in \dot{P}_k$ we define $\tilde{p} := p$. For $p \notin \dot{P}_k$, $\tilde{p}$ is a modification of p in such a way that its outputs are consistent with the current history $\dot{y}\dot{x}_{<k}$, hence $\tilde{p} \in \dot{P}_k$, but unaltered for the current and future cycles $\geq k$. Using this definition of V_{km} we could take the maximium over all policies p in (21), rather than only the consistent ones.

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[9]Here, and everywhere else, with $\xi_k \to \mu_k$ we mean $\xi_k - \mu_k \to 0$, and not that μ_k (and ξ_k) itself converge to a limiting value.

Definition 10 (Intelligence order relation) *We call a policy p more or equally intelligent than p' and write*

$$p \succeq p' \quad :\Leftrightarrow \quad \forall k \forall \ddot{y}\ddot{x}_{<k} : V_{km_k}^{p\xi}(\ddot{y}\ddot{x}_{<k}) \geq V_{km_k}^{p'\xi}(\ddot{y}\ddot{x}_{<k}).$$

i.e. if p yields in any circumstance higher ξ-expected reward than p'.

As the algorithm p^* behind the AIξ agent maximizes $V_{km_k}^{p\xi}$ we have $p^\xi \succeq p$ for all p. The AIξ model is hence the most intelligent agent w.r.t. $\succeq$. $\succeq$ is a universal order relation in the sense that it is free of any parameters (except m_k) or specific assumptions about the environment. A proof, that $\succeq$ is a reliable intelligence order (what we believe to be true), would prove that AIξ is universally optimal. We could further ask: how useful is $\succeq$ for ordering policies of practical interest with intermediate intelligence, or how can $\succeq$ help to guide toward constructing more intelligent systems with reasonable computation time. An effective intelligence order relation $\succeq^c$ will be defined in Section 6, which is more useful from a practical point of view.

4.2   On the Optimality of AIXI

In this section we outline ways towards an optimality proof of AIXI. Sources of inspiration are the SP loss bounds proven in Section 3 and optimality criteria from the adaptive control literature (mainly) for linear systems [KV86]. The value bounds for AIXI are expected to be, in a sense, weaker than the SP loss bounds because the problem class covered by AIXI is much larger than the class of induction problems. Convergence of ξ to μ has already been proven, but is not sufficient to establish convergence of the behavior of the AIXI model to the behavior of the AIμ model. We will focus on three approaches towards a general optimality proof:

What is meant by universal optimality. The first step is to investigate what we can expect from AIXI, i.e. what is meant by *universal optimality*. A "learner" (like AIXI) may converge to the optimal informed decision maker (like AIμ) in several senses. Possibly relevant concepts from statistics are, *consistency, self-tuningness, self-optimizingness, efficiency, unbiasedness, asymptotic* or *finite* convergence [KV86], Pareto-optimality, and some more defined in Scetion 4.3. Some concepts are stronger than necessary, others are weaker than desirable but suitable to start with. Self-optimizingness is defined as the asymptotic convergence of the average true value $\frac{1}{m}V_{1m}^{p^\xi\mu}$ of AIξ to the optimal value $\frac{1}{m}V_{1m}^{*\mu}$. Apart from convergence-speed, self-optimizingness of AIXI would most closely correspond to the loss bounds proven for SP. We investigate which properties are desirable and under which circumstances the AIXI model satisfies these properties. We will show that no universal model, including AIXI, can in general be self-optimizing. On the other hand we show that AIXI is Pareto-optimal in the sense that there is no other policy which performs better or equal in all environments, and strictly better in at least one.

Limited environmental classes. The problem of defining and proving general value bounds becomes more feasible by considering, in a first step, restricted concept classes.

We analyze AIXI for known classes (like Markovian or factorizable environments) and especially for the new classes (forgetful, relevant, asymptotically learnable, farsighted, uniform and (pseudo-)passive) defined later in Section 4.3.

Generalization of AIXI to general Bayes-mixtures. The other approach is to generalize AIXI to AIζ, where $\zeta() = \sum_{\nu \in \mathcal{M}} w_\nu \nu()$ is a general Bayes-mixture of distributions ν in some class $\mathcal{M}$. If $\mathcal{M}$ is the multi-set of enumerable semi-measures enumerated by a Turing-machine, then AIζ coincides with AIXI. If $\mathcal{M}$ is the (multi)set of passive effective environments, then AIXI reduces to the Λ_ξ predictor which has been shown to perform well. One can show that these loss/value bounds generalize to wider classes, at least asymptotically [Hut02c]. Promising classes, are again, the ones described in Section 4.3. Especially for ergodic MDPs we showed that AIζ is self-optimizing. Obviously, the least we must demand from $\mathcal{M}$ to have a chance of finding a self-optimizing policy is that there exists some self-optimizing policy at all. The key result in [Hut02c] is that this necessary condition is sufficient. More generally, the key is not to prove absolute results for specific problem classes, but to prove relative results of the form "if there exists a policy with certain desirable properties, then AIζ also possesses these desirable properties". If there are tasks which cannot be solved by any policy, AIζ cannot be blamed for failing. Environmental classes which allow for self-optimizing policies include bandits, i.i.d. processes, classification tasks, certain classes of POMDPs, k^{th} order ergodic MDPs, factorizable environments, repeated games, and prediction problems. Note that in this approach we have for each environmental class a corresponding model AIζ, whereas in the approach pursued in this article the same universal AIXI model is analyzed for all environmental classes.

4.3 Value Bounds and Separability Concepts

Introduction. The values V_{km} associated with the AI systems correspond roughly to the negative loss $-L_{n\rho}$ of the SP systems. In SP, we were interested in small bounds for the loss excess $L_n^{\Lambda_\xi} - L_n^{\Lambda}$. Unfortunately, simple value bounds for AIξ in terms of V_{km} analogous to the loss bound (20) do not hold. We even have difficulties in specifying what we can expect to hold for AIξ or any AI system which claims to be universally optimal. Consequently, we cannot have a proof if we don't know what to prove. In SP, the only important property of μ for proving error bounds was its complexity $K(\mu)$. We will see that in the AI case, there are no useful bounds in terms of $K(\mu)$ only. We either have to study restricted problem classes or consider bounds depending on other properties of μ, rather than on its complexity only. In the following, we will exhibit the difficulties by two examples and introduce concepts which may be useful for proving value bounds. Despite the difficulties in even claiming useful value bounds, we nevertheless, firmly believe that the order relation (10) correctly formalizes the intuitive meaning of intelligence and, hence, that the AIξ agent is universally optimal.

(Pseudo) Passive μ and the HeavenHell example. In the following, we choose $m_k = m$. We want to compare the true, i.e. μ-expected value V_{1m}^μ of a μ independent

universal policy p^{best} with any other policy p. Naively, we might expect the existence of a policy p^{best} which maximizes V_{1m}^{μ}, apart from additive corrections of lower order for $m \to \infty$

$$V_{1m}^{p^{best}\mu} \; \geq \; V_{1m}^{p\mu} - o(...) \quad \forall \mu, p \tag{26}$$

Such policies are sometimes called self-optimizing [KV86]. Note, that $V_{1m}^{p^{\mu}\mu} \geq V_{1m}^{p\mu} \, \forall p$, but p^{μ} is not a candidate for (a universal) p^{best} as it depends on μ. On the other hand, the policy p^{ξ} of the AIξ agent maximizes V_{1m}^{ξ} by definition ($p^{\xi} \succeq p$). As V_{1m}^{ξ} is thought to be a guess of V_{1m}^{μ}, we might expect $p^{best} = p^{\xi}$ to approximately maximize V_{1m}^{μ}, i.e. (26) to hold. Let us consider the problem class (set of environments) $\mathcal{M} = \{\mu_0, \mu_1\}$ with $\mathcal{Y} = \mathcal{R} = \{0,1\}$ and $r_k = \delta_{iy_1}$ in environment μ_i, where the Kronecker symbol δ_{xy} is defined as 1 for $x = y$ and 0 otherwise. The first action y_1 decides whether you go to heaven with all future rewards r_k being 1 (good) or to hell with all future rewards being 0 (bad). It is clear, that if μ_i, i.e. i is known, the optimal policy p^{μ_i} is to output $y_1 = i$ in the first cycle with $V_{1m}^{p^{\mu_i}\mu} = m$. On the other hand, any unbiased policy p^{best} independent of the actual μ either outputs $y_1 = 1$ or $y_1 = 0$. Independent of the actual choice y_1, there is always an environment ($\mu = \mu_{1-y_1}$) for which this choice is catastrophic ($V_{1m}^{p^{best}\mu} = 0$). No single agent can perform well in both environments μ_0 and μ_1. The r.h.s. of (26) equals $m - o(m)$ for $p = p^{\mu}$. For all p^{best} there is a μ for which the l.h.s. is zero. We have shown that no p^{best} can satisfy (26) for all μ and p, so we cannot expect p^{ξ} to do so. Nevertheless, there are problem classes for which (26) holds, for instance SP. For SP, (26) is just a reformulation of (20) with an appropriate choice for p^{best}, namely Λ_{ξ} (which differs from p^{ξ}, see next section). We expect (26) to hold for all inductive problems in which the environment is not influenced[10] by the output of the agent. We want to call these μ, *passive* or *inductive* environments. Further, we want to call $\mathcal{M}$ and $\mu \in \mathcal{M}$ satisfying (26) with $p^{best} = p^{\xi}$ *pseudo passive*. So we expect inductive μ to be pseudo passive.

The OnlyOne example. Let us give a further example to demonstrate the difficulties in establishing value bounds. Let $\mathcal{R} = \{0,1\}$ and $|\mathcal{Y}|$ be large. We consider all (deterministic) environments in which a single complex output y^* is correct ($r=1$) and all others are wrong ($r=0$). The problem class $\mathcal{M}$ is defined by

$$\mathcal{M} := \{\mu : \mu(y\underline{x}_{<k}y_k\underline{1}) = \delta_{y_k y^*} \, \forall k, \; y^* \in \mathcal{Y}, \; K(y^*) = \lfloor \log_2 |\mathcal{Y}| \rfloor\}$$

There are $N \stackrel{\times}{=} |\mathcal{Y}|$ such y^*. The only way a μ-independent policy p can find the correct y^*, is by trying one y after the other in a certain order. In the first $N-1$ cycles at most, $N-1$ different y are tested. As there are N different possible y^*, there is always a $\mu \in \mathcal{M}$ for which p gives erroneous outputs in the first $N-1$ cycles. The number of errors are $E_{\infty}^p \geq N-1 \stackrel{\times}{=} |\mathcal{Y}| \stackrel{\times}{=} 2^{K(y^*)} \stackrel{\times}{=} 2^{K(\mu)}$ for this μ. As this is true for any p, it is also true for the AIξ model, hence $E_k^{p^{\xi}} \leq 2^{K(\mu)}$ is the best possible error bound we can expect, which depends on $K(\mu)$ only. Actually, we will derive such a bound in Section 5.1 for inductive environments. Unfortunately, as we are mainly interested in the cycle

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[10]Of course, the reward feedback r_k depends on the agent's output. What we have in mind is, like in sequence prediction, that the true sequence is not influenced by the agent.

region $k \ll |\mathcal{Y}| \stackrel{\times}{=} 2^{K(\mu)}$ (see Section 2.8) this bound is trivial. There are no interesting bounds for deterministic μ depending on $K(\mu)$ only, unlike the SP case. Bounds must either depend on additional properties of μ or we have to consider specialized bounds for restricted problem classes. The case of probabilistic μ is similar. Whereas for SP there are useful bounds in terms of $L_k^{\Lambda_\mu}$ and $K(\mu)$, there are no such bounds for AIξ. Again, this is not a drawback of AIξ since for no unbiased AI system the errors/rewards could be bound in terms of $K(\mu)$ and the errors/rewards of AIμ only.

There is a way to make use of gross (e.g. $2^{K(\mu)}$) bounds. Assume that after a reasonable number of cycles k, the information $\dot{x}_{<k}$ perceived by the AIξ agent contains a lot of information about the true environment μ. The information in $\dot{x}_{<k}$ might be coded in any form. Let us assume that the complexity $K(\mu|\dot{x}_{<k})$ of μ under the condition that $\dot{x}_{<k}$ is known, is of order 1. Consider a theorem, bounding the sum of rewards or of other quantities over cycles $1...\infty$ in terms of $f(K(\mu))$ for a function f with $f(O(1)) = O(1)$, like $f(n) = 2^n$. Then, there will be a bound for cycles $k...\infty$ in terms of $\approx f(K(\mu|\dot{x}_{<k})) = O(1)$. Hence, a bound like $2^{K(\mu)}$ can be replaced by small bound $\approx 2^{K(\mu|\dot{x}_{<k})} = O(1)$ after k cycles. All one has to show/ensure/assume is that enough information about μ is presented (in any form) in the first k cycles. In this way, even a gross bound could become useful. In Section 5.4 we use a similar argument to prove that AIξ is able to learn supervised.

Asymptotic learnability. In the following, we weaken (26) in the hope of getting a bound applicable to wider problem classes than the passive one. Consider the I/O sequence $\dot{y}_1\dot{x}_1...\dot{y}_n\dot{x}_n$ caused by AIξ. On history $\ddot{y}\ddot{x}_{<k}$, AIξ will output $\dot{y}_k \equiv \dot{y}_k^\xi$ in cycle k. Let us compare this to $\dot{y}_k^\mu$ what AIμ would output, still on the same history $\dot{y}\dot{x}_{<k}$ produced by AIξ. As AIμ maximizes the μ-expected value, AIξ causes lower (or at best equal) $V_{km_k}^\mu$, if $\dot{y}_k^\xi$ differs from $\dot{y}_k^\mu$. Let $D_{n\mu\xi} := \langle \sum_{k=1}^n 1 - \delta_{\dot{y}_k^\mu, \dot{y}_k^\xi} \rangle_\mu$ be the μ-expected number of suboptimal choices of AIξ, i.e. outputs different from AIμ in the first n cycles. One might weigh the deviating cases by their severity. Especially when the μ-expected rewards $V_{km_k}^{p\mu}$ for $\dot{y}_k^\xi$ and $\dot{y}_k^\mu$ are equal or close to each other, this should be taken into account in a definition of $D_{n\mu\xi}$, e.g. by a weight factor $[V_{km}^{*\mu}(y\!x_{<k}) - V_{km}^{p\xi\mu}(y\!x_{<k})]$. These details do not matter in the following qualitative discussion. The important difference to (26) is that here we stick on the history produced by AIξ and count a wrong decision as, at most, one error. The wrong decision in the HeavenHell example in the first cycle no longer counts as losing m rewards, but counts as one wrong decision. In a sense, this is fairer. One shouldn't blame somebody too much who makes a single wrong decision for which he just has too little information available, in order to make a correct decision. The AIξ model would deserve to be called asymptotically optimal, if the probability of making a wrong decision tends to zero, i.e. if

$$D_{n\mu\xi}/n \to 0 \quad \text{for} \quad n \to \infty, \quad \text{i.e.} \quad D_{n\mu\xi} = o(n). \tag{27}$$

We say that μ can be *asymptotically learned* (by AIξ) if (27) is satisfied. We claim that AIξ (for $m_k \to \infty$) can asymptotically learn every problem μ of relevance, i.e. AIξ is asymptotically optimal. We included the qualifier *of relevance*, as we are not sure whether there could be strange μ spoiling (27) but we expect those μ to be irrelevant from

the perspective of AI. In the field of Learning, there are many asymptotic learnability theorems, often not too difficult to prove. So a proof of (27) might also be feasible. Unfortunately, asymptotic learnability theorems are often too weak to be useful from a practical point. Nevertheless, they point in the right direction.

Uniform μ. From the convergence (24) of $\mu \to \xi$ we might expect $V_{km_k}^{p\xi} \to V_{km_k}^{p\mu}$ and hence, $\dot{y}_k^{\xi}$ defined in (23) to converge to $\dot{y}_k^{\mu}$ defined in (12) with μ probability 1 for $k \to \infty$. The first problem is, that if the V_{km_k} for the different choices of y_k are nearly equal, then even if $V_{km_k}^{p\xi} \approx V_{km_k}^{p\mu}$, $\dot{y}_k^{\xi} \neq \dot{y}_k^{\mu}$ is possible due to the non-continuity of argmax_{y_k}. This can be cured by a weighted $D_{n\mu\xi}$ as described above. More serious is the second problem we explain for $h_k = 1$ and $\mathcal{X} = \mathcal{R} = \{0,1\}$. For $\dot{y}_k^{\xi} \equiv \mathrm{argmax}_{y_k} \xi(\ddot{y}\dot{r}_{<k} y_k \underline{1})$ to converge to $\dot{y}_k^{\mu} \equiv \mathrm{argmax}_{y_k} \mu(\ddot{y}\dot{r}_{<k} y_k \underline{1})$, it is not sufficient to know that $\xi(\ddot{y}\dot{r}_{<k} \ddot{y}\underline{\dot{r}}_k) \to \mu(\ddot{y}\dot{r}_{<k} \ddot{y}\underline{\dot{r}}_k)$ as proven in (24). We need convergence not only for the true output $\dot{y}_k$ and reward $\dot{r}_k$, but also for alternate outputs y_k and reward 1. $\dot{y}_k^{\xi}$ converges to $\dot{y}_k^{\mu}$ if ξ converges uniformly to μ, i.e. if in addition to (24)

$$|\mu(y x_{<k} y_k' \underline{x}_k') - \xi(y x_{<k} y_k' \underline{x}_k')| \; < \; c \cdot |\mu(y x_{<k} y \underline{x}_k) - \xi(y x_{<k} y \underline{x}_k)| \quad \forall y_k' x_k' \qquad (28)$$

holds for some constant c (at least in a μ-expected sense). We call μ satisfying (28) *uniform*. For uniform μ one can show (27) with appropriately weighted $D_{n\mu\xi}$ and bounded horizon $h_k < h_{max}$. Unfortunately there are relevant μ which are not uniform.

Other concepts. In the following, we briefly mention some further concepts. A *Markovian* μ is defined as depending only on the last cycle, i.e. $\mu(y x_{<k} y \underline{x}_k) = \mu_k(x_{k-1} y \underline{x}_k)$. We say μ is *generalized (l^{th} order) Markovian*, if $\mu(y x_{<k} y \underline{x}_k) = \mu_k(x_{k-l} y x_{k-l+1:k-1} y \underline{x}_k)$ for fixed l. This property has some similarities to *factorizable* μ defined in (15). If further $\mu_k \equiv \mu_1 \forall k$, μ is called *stationary*. Further, we call μ (ξ) *forgetful* if $\mu(y x_{<k} y \underline{x}_k)$ ($\xi(y x_{<k} y \underline{x}_k)$) become(s) independent of $y x_{<l}$ for fixed l and $k \to \infty$ with μ probability 1. Further, we say μ is *farsighted*, if $\lim_{m_k \to \infty} \dot{y}_k^{(m_k)}$ exists. More details will be given in Section 4.5, where we also give an example of a farsighted μ for which nevertheless the limit $m_k \to \infty$ makes no sense.

Summary. We have introduced several concepts, which might be useful for proving value bounds, including forgetful, relevant, asymptotically learnable, farsighted, uniform, (generalized) Markovian, factorizable and (pseudo) passive μ. We have sorted them here, approximately in the order of decreasing generality. We want to call them *separability concepts*. The more general (like relevant, asymptotically learnable and farsighted) μ will be called weakly separable, the more restrictive (like (pseudo) passive and factorizable) μ will be called strongly separable, but we will use these qualifiers in a more qualitative, rather than rigid sense. Other (non-separability) concepts are deterministic μ and, of course, the class of all chronological μ.

4.4 Pareto Optimality of AIξ

This subsection shows Pareto-opimtality of AIξ analogous to SP. The total μ-expected reward $V_\mu^{p\xi}$ of policy p^ξ of the AIξ model is of central interest in judging the performance

of AIξ. We know that there *are* policies (e.g. p^μ of AIμ) with higher μ-value ($V_\mu^* \geq V_\mu^{p^\xi}$). In general, every policy based on an estimate ρ of μ which is closer to μ than ξ is, outperforms p^ξ in environment μ, simply because it is more taylored toward μ. On the other hand, such a system probably performs worse than p^ξ in other environments. Since we do not know μ in advance we may ask whether there exists a policy p with better or equal performance than p^ξ in *all* environments $\nu \in \mathcal{M}$ and a strictly better performance for one $\nu \in \mathcal{M}$. This would clearly render p^ξ suboptimal. We show that there is no such p.

Definition 11 (Pareto Optimality) *A policy $\tilde{p}$ is called Pareto-optimal if there is no other policy p with $V_\nu^p \geq V_\nu^{\tilde{p}}$ for all $\nu \in \mathcal{M}$ and strict inequality for at least one ν.*

Theorem 12 (Pareto Optimality) *AIξ alias p^ξ is Pareto-optimal.*

Pareto-optimality should be regarded as a necessary condition for an agent aiming to be optimal. From a practical point of view a significant increase of V for many environments ν may be desirable even if this causes a small decrease of V for a few other ν. The impossibility of such a "balanced" improvement is a more demanding condition on p^ξ than pure Pareto-optimality. In [Hut02c] it has been shown that AIξ is also balanced-Pareto-optimal.

4.5   The Choice of the Horizon

The only significant arbitrariness in the AIξ model lies in the choice of the horizon function $h_k \equiv m_k - k + 1$. We discuss some choices which seem to be natural and give preliminary conclusions at the end. We will not discuss ad hoc choices of h_k for specific problems (like the discussion in Section 5.2 of Section 5 in the context of finite strategic games). We are interested in universal choices of m_k.

Fixed horizon. If the lifetime of the agent is known to be m, which is in practice always large but finite, then the choice $m_k = m$ maximizes correctly the expected future reward. m is usually not known in advance, as in many cases the time we are willing to run a agent depends on the quality of its outputs. For this reason, it is often desirable that good outputs are not delayed too much, if this results in a marginal reward increase only. This can be incorporated by damping the future rewards. If, for instance, we assume that the survival of the agent in each cycle is proportional to the past reward an exponential damping $r_k := r_k' \cdot e^{-\lambda k}$ is appropriate, where r_k' are bounded, e.g. $r_k' \in [0,1]$. The expression (23) converges for $m_k \to \infty$ in this case[11]. But this does not solve the problem, as we introduced a new arbitrary time-scale $1/\lambda$. Every damping introduces a time-scale. Taking $\lambda \to 0$ is prone to the same problems as $m_k \to \infty$ in the undiscounted case discussed below.

Dynamic horizon (universal & harmonic discounting). The largest enumerable horizon with guaranteed finite reward sum can be obtained by the discount $r_k \rightsquigarrow r_k \cdot 2^{-K(k)}$.

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[11]More precisely $\dot{y}_k = \arg\max\limits_{y_k} \lim\limits_{m_k \to \infty} V_{km_k}^{*\xi}(\dot{y}\dot{x}_{<k}y_k)$ exists.

This is the (currently) most attractive universal discount. It is similar to a near-harmonic discount $r_k \rightsquigarrow r_k \cdot k^{-(1+\varepsilon)}$, since $2^{-K(k)} \leq 1/k$ for most k and $2^{-K(k)} \geq c/(k \log^2 k)$. More generally, the time-scale invariant damping factor $r_k = r'_k \cdot k^{-\alpha}$ introduces a dynamic time-scale. In cycle k the contribution of cycle $2^{1/\alpha} \cdot k$ is damped by a factor $\frac{1}{2}$. The effective horizon h_k in this case is $\sim k$. The choice $h_k = \beta \cdot k$ with $\beta \sim 2^{1/\alpha}$ qualitatively models the same behavior. We have not introduced an arbitrary time-scale m, but limited the farsightedness to some multiple (or fraction) of the length of the current history. This avoids the pre-selection of a global time-scale m or $1/\lambda$. This choice has some appeal, as it seems that humans of age k years usually do not plan their lives for more than, perhaps, the next k years ($\beta_{human} \approx 1$). From a practical point of view this model might serve all needs, but from a theoretical point we feel uncomfortable with such a limitation in the horizon from the very beginning. Note, that we have to choose $\beta = O(1)$ because otherwise we would again introduce a number β, which has to be justified.

Infinite horizon (a mathematical solution?). The naive limit $m_k \rightarrow \infty$ in (23) may turn out to be well defined and the previous discussion superfluous. In the following, we suggest a limit which is always well defined (for finite $\mathcal{Y}$). Let $\dot{y}_k^{(m_k)}$ be defined as in (23) with dependence on m_k made explicit. Further, let $\dot{\mathcal{Y}}_k^{(m)} := \{\dot{y}_k^{(m_k)} : m_k \geq m\}$ be the set of outputs in cycle k for the choices $m_k = m, m+1, m+2, \dots$. Because $\dot{\mathcal{Y}}_k^{(m)} \supseteq \dot{\mathcal{Y}}_k^{(m+1)} \neq \{\}$, we have $\dot{\mathcal{Y}}_k^{(\infty)} := \bigcap_{m=k}^{\infty} \dot{\mathcal{Y}}_k^{(m)} \neq \{\}$. We define the $m_k = \infty$ model to output any $\dot{y}_k^{(\infty)} \in \dot{\mathcal{Y}}_k^{(\infty)}$. This is the best output consistent with some arbitrary large choice of m_k. Choosing the lexicographically smallest $\dot{y}_k^{(\infty)} \in \dot{\mathcal{Y}}_k^{(\infty)}$ would correspond to the limes inferior $\underline{\lim}_{m \rightarrow \infty} \dot{y}_k^{(m)}$, which always exists (for finite $\mathcal{Y}$). Generally $\dot{y}_k^{(\infty)} \in \dot{\mathcal{Y}}_k^{(\infty)}$ is unique, i.e. $|\dot{\mathcal{Y}}_k^{(\infty)}| = 1$ iff the naive limit $\lim_{m \rightarrow \infty} \dot{y}_k^{(m)}$ exists. Note, that the limit $\lim_{m \rightarrow \infty} V_{km}^*(yx_{<k})$ needs not to exist for this construction.

Average reward and differential gain. Taking the raw average reward $(r_k + \dots + r_m)/(m-k+1)$ and $m \rightarrow \infty$ also does not help: take an arbitrary policy for the first k time steps and the/an optimal policy for the remaining steps $k+1\dots\infty$. All these policies give the same average. In MDP environments with a single recurrent class one can define the relative or differential gain [Ber95]. In more general environments (we are interested in) the differential gain can be infinite, which is acceptable, since differential gains can still be totally ordered. The major problem is the *existence* of the differential gain, i.e. whether it converges for $m \rightarrow \infty$ in $\mathbb{R} \cup \{\infty\}$ at all (and does not oscillate). This is just the old convergence problem in slightly different form.

Immortal agents are lazy. The construction above leads to a mathematically elegant, no-parameter AIξ model. Unfortunately this is not the end of the story. The limit $m_k \rightarrow \infty$ can cause undesirable results in the AIμ model for special μ, which might also happen in the AIξ model whatever we define $m_k \rightarrow \infty$. Consider $\mathcal{Y} = \mathcal{X} = \mathcal{R} = \{0,1\}$. Output $y_k = 0$ shall give reward $r_k = 0$ and output $y_k = 1$ shall give $r_k = 1$ iff $\dot{y}_{k-l-\sqrt{l}} \dots \dot{y}_{k-l} = 0\dots0$ for some l. I.e. the agent can achieve l consecutive positive rewards if there was a sequence of length at least $\sqrt{l}$ with $y_k = r_k = 0$. If the lifetime of the AIμ agent is m, it outputs $\dot{y}_k = 0$ in the first r cycles and then $\dot{y}_k = 1$ for the remaining r^2 cycles with r such that

$r + r^2 = m$. This will lead to the highest possible total reward $V_{1m} = r^2 = m + \frac{1}{2} - \sqrt{m + \frac{1}{4}}$. Any fragmentation of the 0 and 1 sequences would reduce V_{1m}. For $m \to \infty$ the AIμ agent can and will delay the point r of switching to $\dot{y}_k = 1$ indefinitely and always output 0 leading to total reward 0, obviously the worst possible behavior. The AIξ agent will explore the above rule after a while of trying $y_k = 0/1$ and then applies the same behavior as the AIμ agent, since the simplest rules covering past data dominate ξ. For finite m this is exactly what we want, but for infinite m the AIξ model (probably) fails just as the AIμ model does. The good point is, that this is not a weakness of the AIξ model, as AIμ fails too and no agent can be better than AIμ. The bad point is that $m_k \to \infty$ has far reaching consequences, even when starting from an already very large $m_k = m$. The reason being that the μ of this example is highly non-local in time, i.e. it may violate one of our weak separability conditions.

Convergence of AIξ to AIμ for infinite horizon. In the last paragraph we have considered the consequences of $m_k \to \infty$ in the AIμ model. We now consider whether the AIξ model is a good approximation of the AIμ model for large m_k. Another objection against too large choices of m_k is that $\xi(y\underline{x}_{<k}y\underline{x}_{k:m_k})$ has been proven to be a good approximation of $\mu(y\underline{x}_{<k}y\underline{x}_{k:m_k})$ only for $k \gg h_k$, which is never satisfied for $m_k = m \to \infty$. We have seen that, for factorizable μ, the limit $h_k \to \infty$ causes no problem, as from a certain h_k on the output $\dot{y}_k$ is independent of h_k. As $\xi \to \mu$ for bounded h_k, ξ will develop this separability property too. So, from a certain k_0 on the limit $h_k \to \infty$ might also be safe for ξ if μ is factorizable. Therefore, taking the limit $h_k \to \infty$ for all k worsens the behavior of AIξ maybe only for finitely many cycles $k \le k_0$, which would be acceptable. We suppose that the rewards $r_{k'}$ for $k' \gg k$, where ξ can no longer be trusted as a good approximation to μ, are in some sense randomly disturbed with decreasing influence on the choice of $\dot{y}_k$. This claim is supported by the forgetfulness property of ξ.

Conclusions. We are not sure whether the choice of m_k is of marginal importance, as long as m_k is chosen sufficiently large and of low complexity, $m_k = 2^{2^{16}}$ for instance, or whether the choice of m_k will turn out to be a central topic for the AIξ model or for the planning aspect of any AI system in general. We suppose that the limit $m_k \to \infty$ for the AIξ model results in correct behavior for weakly separable μ. A proof of this conjecture, if true, would probably give interesting insights.

4.6   Outlook

Expert advice approach. We considered expected performance bounds for predictions based on Solomonoff's prior. The other, dual, currently very popular approach, is "prediction with expert advice" (WM) invented by Littlestone and Warmuth (1989), Vovk (1992). Whereas WM performs well in any environment, but only relative to a given set of experts , our Λ_ξ predictor competes with *any* other predictor, but only in expectation for environments with computable distribution. It seems philosophically less compromising to make assumptions on prediction strategies than on the environment, however weak. One could investigate whether WM can be generalized to the case of active agents, which

would result in a model dual to AIXI. We believe the answer to be negative, which on the positive side would show the necessity of Occam's razor assumption, and the distinguishedness of AIXI.

Actions as random variables. The uniqueness for the choice of the generalized ξ (17) in the AIXI model could be explored. From the originally many alternatives, which could all be ruled out, there is one alternative which still seems possible. Instead of defining ξ as in (22) one could treat the agent's actions y also as universally distributed random variables and then conditionalize ξ on y by Bayes' rule.

Structure of AIXI. The algebraic properties and the structure of AIXI could be investigated in more depth. This would extract the essentials from AIXI which finally could lead to an axiomatic characterization of AIXI. The benefit is as in any axiomatic approach. It would clearly exhibit the assumptions, separate the essentials from technicalities, and simplify understanding and, most important, guide in finding proofs.

Restricted policy classes. The development in this section could be scaled down to restricted classes of policies $\mathcal{P}$. One may define $V^* = \mathrm{argmax}_{p \in \mathcal{P}} V^p$. For instance, consider a finite class of quickly computable policies. For MDPs, ξ is quickly computable and V_ξ^p can be (efficiently) computed by Monte-Carlo sampling. Maximizing over the finitely many policies $p \in \mathcal{P}$ selects the asymptotically best policy p^ξ from $\mathcal{P}$ for all (ergodic) MDPs [Hut02c].

4.7 Conclusions

All tasks which require intelligence to be solved can naturally be formulated as a maximization of some expected utility in the framework of agents. We gave an explicit expression (12) of such a decision theoretic agent. The main remaining problem is the unknown prior probability distribution μ^{AI} of the environment(s). Conventional learning algorithms are unsuitable, because they can neither handle large (unstructured) state spaces, nor do they converge in the theoretically minimal number of cycles, nor can they handle non-stationary environments appropriately. On the other hand, the universal semimeasure ξ (17), based on ideas from algorithmic information theory, solves the problem of the unknown prior distribution for induction problems. No explicit learning procedure is necessary, as ξ automatically converges to μ. We unified the theory of universal sequence prediction with the decision theoretic agent by replacing the unknown true prior μ^{AI} by an appropriately generalized universal semimeasure ξ^{AI}. We gave strong arguments that the resulting AIξ model is universally optimal. Furthermore, possible solutions to the horizon problem have been discussed. In Section 5 we outline for a number of problem classes, how the AIξ model can solve them. They include sequence prediction, strategic games, function minimization and, especially, how AIξ learns to learn supervised. In Section 6 we develop a more elegant but equivalent functional form of the AIξ model. It is used to define a universal intelligence order relation, to discuss in which sense AIξ is the most intelligent agent, and to construct a modified time-bounded (computable) AIξ^{tl} version.

5   Important Environmental Classes

In order to give further support for the universality and optimality of the AIξ theory, we apply AIξ in this section to a number of problem classes. They include sequence prediction, strategic games, function minimization and, especially, how AIξ learns to learn supervised. For some classes we give concrete examples to illuminate the scope of the problem class. We first formulate each problem class in its natural way (when μ^{problem} is known) and then construct a formulation within the AIμ model and prove its equivalence. We then consider the consequences of replacing μ by ξ. The main goal is to understand why and how the problems are solved by AIξ. We only highlight special aspects of each problem class. Sections 5.1–5.5 together should give a better picture of the AIξ model. We do not study every aspect for every problem class. The subsections might be read selectively. They are not essential to understand the remaining sections.

5.1   Sequence Prediction (SP)

We have introduced the AIξ model as a unification of ideas of sequential decision theory and universal probability distribution. We might expect AIξ to behave identically to SPξ, when faced with a sequence prediction problem, but things are not that simple, as we will see.

Using the AIμ model for sequence prediction. We have seen in the last section how to predict sequences for known and unknown prior distribution μ^{SP}. Here we consider binary sequences[12] $z_1 z_2 z_3... \in I\!B^\infty$ with known prior probability $\mu^{SP}(\underline{z_1 z_2 z_3...})$.

   We want to show how the AIμ model can be used for sequence prediction. We will see that it makes the same prediction as the SPμ agent. For simplicity we only discuss the special error-loss $\ell_{xy} = 1 - \delta_{xy}$, where δ is the Kronecker symbol, defined as $\delta_{ab} = 1$ for $a = b$ and 0 otherwise. First, we have to specify *how* the AIμ model should be used for sequence prediction. The following choice is natural:

   The system's output y_k is interpreted as a prediction for the kth bit z_k of the string under consideration. This means that y_k is binary ($y_k \in I\!B =: \mathcal{Y}$). As a reaction of the environment, the agent receives reward $r_k = 1$ if the prediction was correct ($y_k = z_k$), or $r_k = 0$ if the prediction was erroneous ($y_k \neq z_k$). The question is what the input x'_k in the next cycle should be. One choice would be to inform the agent about the correct k^{th} bit of the string and set $x'_k = z_k$. But as from the reward r_k in conjunction with the prediction y_k, the true bit $z_k = \delta_{y_k r_k}$ can be inferred, this information is redundant. There is no need for this additional feedback. So we set $x'_k = \epsilon \in \mathcal{X} = \{\epsilon\}$ thus having $x_k \equiv r_k$. The agent's performance does not change when we include this redundant information, it merely complicates the notation. The prior probability μ^{AI} of the AIμ model is

$$\mu^{AI}(y_1 \underline{x}_1 ... y_k \underline{x}_k) \;=\; \mu^{AI}(y_1 \underline{r}_1 ... y_k \underline{r}_k) \;=\; \mu^{SP}(\underline{\delta_{y_1 r_1} ... \delta_{y_k r_k}}) \;=\; \mu^{SP}(\underline{z_1 ... z_k}) \qquad (29)$$

In the following, we will drop the superscripts of μ because they are clear from the arguments of μ and the μ equal in any case. It is intuitively clear and can formally be

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[12]We use z_k to avoid notational conflicts with the agent's inputs x_k.

shown [Hut00] that maximizing the future reward V_{km}^μ is identical to greedily maximizing the immediate expected reward V_{kk}^μ. There is no exploration-exploitation tradeoff in the prediction case. Hence, AIμ acts with

$$\dot{y}_k \;=\; \arg\max_{y_k} V_{kk}^{*\mu}(\ddot{y}\ddot{x}_{<k}y_k) \;=\; \arg\max_{y_k} \sum_{r_k} r_k \cdot \mu^{AI}(\ddot{y}\ddot{r}_{<k}y\underline{r}_k) \;=\; \arg\max_{z_k} \mu^{SP}(\dot{z}_1...\dot{z}_{k-1}\underline{z}_k) \tag{30}$$

The first equation is the definition of the agent's prediction (11) with m_k replaced by k. In the second equation we used the definition (10) of V_{km}. In the last equation we used (29) and $r_k = \delta_{y_k z_k}$.

So, the AIμ model predicts that z_k that has maximal μ probability, given $\dot{z}_1...\dot{z}_{k-1}$. This prediction is independent of the choice of m_k. It is exactly the prediction scheme of the deterministic sequence prediction with known prior SPμ described in Section 3.5 (with special error-loss). As this model was optimal, AIμ is optimal too, i.e. has minimal number of expected errors (maximal μ-expected reward) as compared to any other sequence prediction scheme. From this, it is clear that the value $V_{km}^{*\mu}$ must be closely related to the expected loss $L_m^{\Lambda_\mu}$ (19). Indeed one can show that $V_{1m}^{*\mu} = m - L_{m\Lambda_\mu}$, which is also valid for general loss functions.

Using the AIξ model for sequence prediction. Now we want to use the universal AIξ model instead of AIμ for sequence prediction and try to derive error/loss bounds analogous to (20). Like in the AIμ case, the agent's output y_k in cycle k is interpreted as a prediction for the kth bit z_k of the string under consideration. The reward is $r_k = \delta_{y_k z_k}$ and there are no other inputs $x_k = \epsilon$. What makes the analysis more difficult is that ξ is not symmetric in $y_i r_i \leftrightarrow (1-y_i)(1-r_i)$ and (29) does not hold for ξ. On the other hand, ξ^{AI} converges to μ^{AI} in the limit (24), and (29) should hold asymptotically for ξ in some sense. So we expect that everything proven for AIμ holds approximately for AIξ. The AIξ model should behave similarly to Solomonoff prediction SPξ. Especially we expect error bounds similar to (20). Making this rigorous seems difficult. Some general remarks have been made in the last section. Note that bounds like (26) can't hold in general, but could be valid for AIξ in (pseudo) passive environments.

Here we concentrate on the special case of a deterministic computable environment, i.e. the environment is a sequence $\dot{z} = \dot{z}_1\dot{z}_2...$ with $K(\dot{z}_{1:\infty}) < \infty$. Furthermore, we only consider the simplest horizon model $m_k = k$, i.e. maximize only the next reward. This is sufficient for sequence prediction, as the reward of cycle k only depends on output y_k and not on earlier decisions. This choice is in no way sufficient and satisfactory for the full AIξ model, as *one* single choice of m_k should serve for *all* AI problem classes. So AIξ should allow good sequence prediction for some universal choice of m_k and not only for $m_k = k$, which definitely does not suffice for more complicated AI problems. The analysis of this general case is a challenge for the future. For $m_k = k$ the AIξ model (23) with $x_i' = \varepsilon$ reduces to

$$\dot{y}_k \;=\; \arg\max_{y_k} \sum_{r_k} r_k \cdot \xi(\ddot{y}\ddot{r}_{<k}y\underline{r}_k) \;=\; \arg\max_{y_k} \xi(\ddot{y}\ddot{r}_{<k}y_k\underline{1}) \tag{31}$$

The environmental response $\dot{r}_k$ is given by $\delta_{\dot{y}_k \dot{z}_k}$; it is 1 for a correct prediction ($\dot{y}_k = \dot{z}_k$) and 0 otherwise. One can show [Hut00] that the number of wrong predictions $E_{n\xi}^{AI}$ of the

AIξ model (31) in these environments is bounded by

$$E_{n\xi}^{AI} \overset{\times}{\leq} 2^{K(\dot{z}_{1:n})} \leq 2^{K(\dot{z}_{1:\infty})} < \infty \tag{32}$$

for a computable deterministic environment string $\dot{z}_1\dot{z}_2...$. The intuitive interpretation is that each wrong prediction eliminates at least one program p of size $l(p) \overset{+}{\leq} K(\dot{z})$. The size is smaller than $K(\dot{z})$, as larger policies could not mislead the agent to a wrong prediction, since there is a program of size $K(\dot{z})$ making a correct prediction. There are at most $2^{K(\dot{z})+O(1)}$ such policies, which bounds the total number of errors.

We have derived a finite bound for $E_{n\xi}^{AI}$, but unfortunately, a rather weak one as compared to (20). The reason for the strong bound in the SP case was that every error eliminates halve of the programs.

The AIξ model would not be sufficient for realistic applications if the bound (32) were sharp, but we have the strong feeling (but currently only weak arguments) that better bounds proportional to $K(\dot{z})$ analogous to (20) exist. The current proof technique is not strong enough for achieving this. One argument for a better bound is the formal similarity between $\operatorname{argmax}_{z_k}\xi(\dot{z}_{<k}\underline{z}_k)$ and (31), the other is that we were unable to construct an example sequence for which AIξ makes more than $O(K(\dot{z}))$ errors.

5.2   Strategic Games (SG)

Introduction. A very important class of problems are strategic games, like chess. In fact, what is subsumed under game theory, is so general, that it includes not only a huge variety of games, from simple games of chance like roulette, combined with strategy like Backgammon, up to purely strategic games like chess or checkers or go. Game theory can also describe political and economic competitions and coalitions, Darwinism and many more. It seems that nearly every AI problem could be brought into the form of a game. Nevertheless, the intention of a game is that several players perform actions with (partial) observable consequences. The goal of each player is to maximize some utility function (e.g. to win the game). The players are assumed to be rational, taking into account all information they posses. The different goals of the players are usually in conflict. For an introduction into game theory, see [FT91, OR94, RN95, NM44].

If we interpret the AI system as one player and the environment models the other rational player *and* the environment provides the reinforcement feedback r_k, we see that the agent-environment configuration satisfies all criteria of a game. On the other hand, the AI system can handle more general situations, since it interacts optimally with an environment, even if the environment is not a rational player with conflicting goals.

Strictly competitive strategic games. In the following, we restrict ourselves to deterministic, strictly competitive strategic[13] games with alternating moves. Player 1 makes move y'_k in round k, followed by the move x'_k of player 2. So a game with n rounds consists

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[13]In game theory, games like chess are often called 'extensive', whereas 'strategic' is reserved for a different kind of game.

of a sequence of alternating moves $y_1'x_1'y_2'x_2'...y_n'x_n'$. At the end of the game in cycle n the game or final board situation is evaluated with $V(y_1'x_1'...y_n'x_n')$. Player 1 tries to maximize V, whereas player 2 tries to minimize V. In the simplest case, V is 1 if player 1 won the game, $V = -1$ if player 2 won and $V = 0$ for a draw. We assume a fixed game length n independent of the actual move sequence. For games with variable length but maximal possible number of moves n, we could add dummy moves and pad the length to n. The optimal strategy (Nash equilibrium) of both players is a minimax strategy

$$\dot{x}_k' = \arg\min_{x_k'} \max_{y_{k+1}'} \min_{x_{k+1}'} ... \max_{y_n'} \min_{x_n'} V(\dot{y}_1'\dot{x}_1'...\dot{y}_k'x_k'...y_n'x_n') \tag{33}$$

$$\dot{y}_k' = \arg\max_{y_k'} \min_{x_k'} ... \max_{y_n'} \min_{x_n'} V(\dot{y}_1'\dot{x}_1'...\dot{y}_{k-1}'\dot{x}_{k-1}'y_k'x_k'...y_n'x_n') \tag{34}$$

But note, that the minimax strategy is only optimal if both players behave rationally. If, for instance, player 2 has limited capabilites or makes errors and player 1 is able to discover these (through past moves) he could exploit these and improve his performance by deviating from the minimax strategy. At least, the classical game theory of Nash equilibria does not take into account limited rationality, whereas the AIξ agent should.

Using the AIμ model for game playing. In the following, we demonstrate the applicability of the AI model to games. The AI system takes the position of player 1. The environment provides the evaluation V. For a symmetric situation we could take a second AI system as player 2, but for simplicity we take the environment as the second player and assume that this environmental player behaves according to the minimax strategy (33). The environment serves as a perfect player *and* as a teacher, albeit a very crude one as it tells the agent at the end of the game, only whether it won or lost.

The minimax behavior of player 2 can be expressed by a (deterministic) probability distribution μ^{SG} as the following

$$\mu^{SG}(y_1'\underline{x}_1'...y_n'\underline{x}_n') := \begin{cases} 1 & \text{if} \quad x_k' = \arg\min_{x_k''} ... \max_{y_n''} \min_{x_n''} V(y_1'x_1'...y_k'x_k''...y_n''x_n'') \ \forall \ k \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

The probability that player 2 makes move x_k' is $\mu^{SG}(\dot{y}_1'\dot{x}_1'...\dot{y}_k'\underline{x}_k')$ which is 1 for $x_k' = \dot{x}_k'$ as defined in (33) and 0 otherwise.

Clearly, the AI system receives no feedback, i.e. $r_1 = ... = r_{n-1} = 0$, until the end of the game, where it should receive positive/negative/neutral feedback on a win/loss/draw, i.e. $r_n = V(...)$. The environmental prior probability is therefore

$$\mu^{AI}(y_1\underline{x}_1...y_n\underline{x}_n) = \begin{cases} \mu^{SG}(y_1'\underline{x}_1'...y_n'\underline{x}_n') & \text{if} \quad r_1...r_{n-1} = 0 \text{ and } r_n = V(y_1'x_1'...y_n'x_n') \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

where $y_i = y_i'$ and $x_i = r_i x_i'$. If the environment is a minimax player (33) plus a crude teacher V, i.e. if μ^{AI} is the true prior probability, the question now is, what is the behavior $\dot{y}_k^{AI}$ of the AIμ agent. It turns out that if we set $m_k = n$ the AIμ agent is also a minimax player (34) and hence optimal ($\dot{y}_k^{AI} = \dot{y}_k^{SG}$, see [Hut00] for a formal proof). Playing a sequence of

games is a special case of a factorizable μ described in Section 2.7 with identical factors μ_r for all r and equal episode lengths $n_{r+1} - n_r = n$.

Hence, in a minimax environment AIμ behaves itself as a minimax strategy,

$$\dot{y}_k^{AI} = \arg\max_{y_k} \min_{x_k'} \ldots \max_{y_{(r+1)n}} \min_{x_{(r+1)n}'} V(\dot{y}\dot{x}_{rn+1:k-1}' \ldots y x_{k:(r+1)n}') \tag{37}$$

with r such that $rn < k \leq (r+1)n$ and for any choice of m_k as long as the horizon $h_k \geq n$.

Using the AIξ model for game playing. When going from the specific AIμ model, where the rules of the game have been explicitly modeled into the prior probability μ^{AI}, to the universal model AIξ we have to ask whether these rules can be learned from the assigned rewards r_k. Here, another (actually the main) reason for studying the case of repeated games, rather than just one game arises. For a single game there is only one cycle of non-trivial feedback namely the end of the game – too late to be useful except when there are further games following.

Even in the case of repeated games, there is only very limited feedback, at most $\log_2 3$ bits of information per game if the 3 outcomes win/loss/draw have the same frequency. So there are at least $O(K(game))$ number of games necessary to learn a game of complexity $K(game)$. Apart from extremely simple games, even this estimate is far too optimistic. As the AIξ agent has no information about the game to begin with, its moves will be more or less random and it can win the first few games merely by pure luck. So the probability that the agent loses is near to one and hence the information content I in the feedback r_k at the end of the game is much less than $\log_2 3$. This situation remains for a very large number of games. But in principle, every game should be learnable after a very long sequence of games even with this minimal feedback only, as long as $I \not\equiv 0$.

The important point is that no other learning scheme with no extra information can learn the game more quickly than AIξ. We expect this to be true as μ^{AI} factorizes in the case of games of fixed length, i.e. μ^{AI} satisfies a strong separability condition. In the case of variable game length the entanglement is also low. μ^{AI} should still be sufficiently separable allowing us to formulate and prove good reward bounds for AIξ.

To learn realistic games like tic-tac-toe (noughts and crosses) in realistic time one has to provide more feedback. This could be achieved by intermediate help during the game. The environment could give positive(negative) feedback for every good(bad) move the agent makes. The demand on whether a move is to be valued as good should be adapted to the gained experience of the agent in such a way that approximately half of the moves are valuated as good and the other half as bad, in order to maximize the information content of the feedback.

For more complicated games like chess, even more feedback is necessary from a practical point of view. One way to increase the feedback far beyond a few bits per cycle is to train the agent by teaching it good moves. This is called supervised learning. Despite the fact that the AI model has only a reward feedback r_k, it is able to learn supervised, as will be shown in Section 5.4. Another way would be to start with more simple games containing certain aspects of the true game and to switch to the true game when the agent has learned the simple game.

No other difficulties are expected when going from μ to ξ. Eventually ξ^{AI} will converge to the minimax strategy μ^{AI}. In the more realistic case, where the environment is not a perfect minimax player, AIξ can detect and exploit the weakness of the opponent.

Finally, we want to comment on the input/output space $\mathcal{X}/\mathcal{Y}$ of the AI system. In practical applications, $\mathcal{Y}$ will possibly include also illegal moves. If $\mathcal{Y}$ is the set of moves of e.g. a robotic arm, the agent could move a wrong figure or even knock over the figures. A simple way to handle illegal moves y_k is by interpreting them as losing moves, which terminate the game. Further, if e.g. the input x_k is the image of a video camera which makes one shot per move, $\mathcal{X}$ is not the set of moves by the environment but includes the set of states of the game board. The discussion in this section handles this case as well. There is no need to explicitly design the systems I/O space $\mathcal{X}/\mathcal{Y}$ for a specific game.

The discussion above on the AIξ agent was rather informal for the following reason: game playing (the SGξ agent) has (nearly) the same complexity as fully general AI, and quantitative results for the AIξ agent are difficult (but not impossible) to obtain.

5.3 Function Minimization (FM)

Applications/examples. There are many problems that can be reduced to a function minimization problem (FM). The minimum of a (real valued) function $f : \mathcal{Y} \to I\!R$ over some domain $\mathcal{Y}$ or a good approximate to the minimum has to be found, usually with some limited resources.

One popular example is the traveling salesman problem (TSP). $\mathcal{Y}$ is the set of different routes between towns and $f(y)$ the length of route $y \in \mathcal{Y}$. The task is to find a route of minimal length visiting all cities. This problem is NP hard. Getting good approximations in limited time is of great importance in various applications. Another example is the minimization of production costs (MPC), e.g. of a car, under several constraints. $\mathcal{Y}$ is the set of all alternative car designs and production methods compatible with the specifications and $f(y)$ the overall cost of alternative $y \in \mathcal{Y}$. A related example is finding materials or (bio)molecules with certain properties (MAT). E.g. solids with minimal electrical resistance or maximally efficient chlorophyll modifications or aromatic molecules that taste as close as possible to strawberry. We can also ask for nice paintings (NPT). $\mathcal{Y}$ is the set of all existing or imaginable paintings and $f(y)$ characterizes how much person A likes painting y. The agent should present paintings, which A likes.

For now, these are enough examples. The TSP is very rigorous from a mathematical point of view, as f, i.e. an algorithm of f, is usually known. In principle, the minimum could be found by exhaustive search, were it not for computational resource limitations. For MPC, f can often be modeled in a reliable and sufficiently accurate way. For MAT you need very accurate physical models, which might be unavailable or too difficult to solve or implement. For NPT all we have is the judgement of person A on every presented painting. The evaluation function f cannot be implemented without scanning $A's$ brain, which is not possible with today's technology.

So there are different limitations, some depending on the application we have in mind. An implementation of f might not be available, f can only be tested at some arguments y

and $f(y)$ is determined by the environment. We want to (approximately) minimize f with as few function calls as possible or, conversely, find an as close as possible approximation for the minimum within a fixed number of function evaluations. If f is available or can quickly be inferred by the agent and evaluation is quick, it is more important to minimize the total time needed to imagine new trial minimum candidates plus the evaluation time for f. As we do not consider computational aspects of AIξ till Section 6 we concentrate on the first case, where f is not available or dominates the computational requirements.

The greedy model. The FM model consists of a sequence $\dot{y}_1\dot{z}_1\dot{y}_2\dot{z}_2...$ where $\dot{y}_k$ is a trial of the FM agent for a minimum of f and $\dot{z}_k = f(\dot{y}_k)$ is the true function value returned by the environment. We randomize the model by assuming a probability distribution $\mu(f)$ over the functions. There are several reasons for doing this. We might really not know the exact function f, as in the NPT example, and model our uncertainty by the probability distribution μ. More importantly, we want to parallel the other AI classes, like in the SPμ model, where we always started with a probability distribution μ that was finally replaced by ξ to get the universal Solomonoff prediction SPξ. We want to do the same thing here. Further, the probabilistic case includes the deterministic case by choosing $\mu(f) = \delta_{ff_0}$, where f_0 is the true function. A final reason is that the deterministic case is trivial when μ and hence f_0 is known, as the agent can internally (virtually) check all function arguments and output the correct minimum from the very beginning.

We will assume that $\mathcal{Y}$ is countable or finite and that μ is a discrete measure, e.g. by taking only computable functions. The probability that the function values of $y_1,...,y_n$ are $z_1,...,z_n$ is then given by

$$\mu^{FM}(y_1\underline{z}_1...y_n\underline{z}_n) := \sum_{f:f(y_i)=z_i \ \forall 1\leq i\leq n} \mu(f) \qquad (38)$$

We start with a model that minimizes the expectation z_k of the function value f for the next output y_k, taking into account previous information:

$$\dot{y}_k := \arg\min_{y_k} \sum_{z_k} z_k \cdot \mu(\dot{y}_1\dot{z}_1...\dot{y}_{k-1}\dot{z}_{k-1}y_k\underline{z}_k)$$

This type of greedy algorithm, just minimizing the next feedback, was sufficient for sequence prediction (SP) and is also sufficient for classification (CF, not described here). It is, however, not sufficient for function minimization as the following example demonstrates.

Take $f:\{0,1\}\rightarrow\{1,2,3,4\}$. There are 16 different functions which shall be equiprobable, $\mu(f) = \frac{1}{16}$. The function expectation in the first cycle

$$\langle z_1 \rangle := \sum_{z_1} z_1 \cdot \mu(y_1\underline{z}_1) = \tfrac{1}{4}\sum_{z_1} z_1 = \tfrac{1}{4}(1+2+3+4) = 2.5$$

is just the arithmetic average of the possible function values and is independent of y_1. Therefore, $\dot{y}_1 = 0$, as argmin is defined to take the lexicographically first minimum in an

ambiguous case. Let us assume that $f_0(0)=2$, where f_0 is the true environment function, i.e. $\dot{z}_1=2$. The expectation of z_2 is then

$$\langle z_2 \rangle \;:=\; \sum_{z_2} z_2 \cdot \mu(02y_2\underline{z}_2) \;=\; \begin{cases} 2 & \text{for} \quad y_2 = 0 \\ 2.5 & \text{for} \quad y_2 = 1 \end{cases}$$

For $y_2 = 0$ the agent already knows $f(0) = 2$, for $y_2 = 1$ the expectation is, again, the arithmetic average. The agent will again output $\dot{y}_2 = 0$ with feedback $\dot{z}_2 = 2$. This will continue forever. The agent is not motivated to explore other $y's$ as $f(0)$ is already smaller than the expectation of $f(1)$. This is obviously not what we want. The greedy model fails. The agent ought to be inventive and try other outputs when given enough time.

The general reason for the failure of the greedy approach is that the information contained in the feedback z_k depends on the output y_k. A FM agent can actively influence the knowledge it receives from the environment by the choice in y_k. It may be more advantageous to first collect certain knowledge about f by an (in greedy sense) non-optimal choice for y_k, rather than to minimize the z_k expectation immediately. The non-minimality of z_k might be over-compensated in the long run by exploiting this knowledge. In SP, the received information is always the current bit of the sequence, independent of what SP predicts for this bit. This is the reason why a greedy strategy in the SP case is already optimal.

The general FMμ/ξ model. To get a useful model we have to think more carefully about what we really want. Should the FM agent output a good minimum in the last output in a limited number of cycles m, or should the average of the $z_1,...,z_m$ values be minimal, or does it suffice that just one of the z is as small as possible? The subtle and important differences between these settings have been analyzed and discussed in detail in [Hut00]. In the following we concentrate on minimizing the average, or equivalently the sum of function values. We define the FMμ model as to minimize the sum $z_1+...+z_m$. Building the μ average by summation over the z_i and minimizing w.r.t. the y_i has to be performed in the correct chronological order. With a similar reasoning as in (8) to (11) we get

$$\dot{y}_k^{FM} \;=\; \arg\min_{y_k} \sum_{z_k} ... \min_{y_m} \sum_{z_m} (z_1 + ... + z_m) \cdot \mu(\dot{y}_1\dot{z}_1...\dot{y}_{k-1}\dot{z}_{k-1}y_k\underline{z}_k...y_m\underline{z}_m) \qquad (39)$$

By construction, the FMμ model guarantees optimal results in the usual sense that no other model knowing only μ can be expected to produce better results. The interesting case (in AI) is when μ is unknown. We define for this case, the FMξ model by replacing $\mu(f)$ with some $\xi(f)$, which should assign high probability to functions f of low complexity. So we might define $\xi(f)=\sum_{q:\forall x[U(qx)=f(x)]}2^{-l(q)}$. The problem with this definition is that it is, in general, undecidable whether a TM q is an implementation of a function f. $\xi(f)$ defined in this way is uncomputable, not even approximable. As we only need a ξ analogous to the l.h.s. of (38), the following definition is natural

$$\xi^{FM}(y_1\underline{z}_1...y_n\underline{z}_n) \;:=\; \sum_{q:q(y_i)=z_i \; \forall 1\leq i\leq n} 2^{-l(q)} \qquad (40)$$

ξ^{FM} is actually equivalent to inserting the uncomputable $\xi(f)$ into (38). One can show that ξ^{FM} is an enumerable semi-measure and dominates all enumerable probability distributions of the form (38).

Alternatively, we could have constrained the sum in (40) by $q(y_1...y_n)=z_1...z_n$ analogous to (22), but these two definitions are not equivalent. Definition (40) ensures the symmetry[14] in its arguments and $\xi^{FM}(...y\underline{z}...y\underline{z}'...)=0$ for $z\neq z'$. It incorporates all general knowledge we have about function minimization, whereas (22) does not. But this extra knowledge has only low information content (complexity of $O(1)$), so we do not expect FMξ to perform much worse when using (22) instead of (40). But there is no reason to deviate from (40) at this point.

We can now define a loss $L_{m\mu}^{FM}$ as (39) with $k=1$ and argmin_{y_1} replaced by $\min_{y_1}$ and, additionally, μ replaced by ξ for $L_{m\xi}^{FM}$. We expect $|L_{m\xi}^{FM}-L_{m\mu}^{FM}|$ to be bounded in a way that justifies the use of ξ instead of μ for computable μ, i.e. computable f_0 in the deterministic case. The arguments are the same as for the AIξ model.

In [Hut00] it has been proven that FMξ is inventive in the sense that it never ceases searching for minima, but will test *all* $y \in \mathcal{Y}$ if $\mathcal{Y}$ is finite (and an infinite set of different $y's$ if $\mathcal{Y}$ is infinite) for sufficiently large horizon m. There are currently no rigorous results on the *quality* of the guesses, but for the FMμ agent the guesses are optimal by definition. If $K(\mu)$ for the true distribution μ is finite, we expect the FMξ agent to solve the 'exploration versus exploitation' problem in a universally optimal way, as ξ converges to μ.

Using the AI models for Function Mininimization. The AI model can be used for function minimization in the following way. The output y_k of cycle k is a guess for a minimum of f, like in the FM model. The reward r_k should be high for small function values $z_k = f(y_k)$. The choice $r_k = -z_k$ for the reward is natural. Here, the feedback is not binary but $r_k \in \mathcal{R} \subset I\!\!R$, with $\mathcal{R}$ being a countable subset of $I\!\!R$, e.g. the computable reals or all rational numbers. The feedback x_k' should be the function value $f(y_k)$. As this is already provided in the rewards r_k we could set $x_k' = \varepsilon$ as in Section 5.1. For a change and to see that the choice really doesn't matter we set $x_k' = z_k$ here. The AIμ prior probability is

$$\mu^{AI}(y_1\underline{x}_1...y_n\underline{x}_n) \;=\; \left\{ \begin{array}{ll} \mu^{FM}(y_1\underline{z}_1...y_n\underline{z}_n) & \text{for } r_k = -z_k, \; x_k' = z_k, \; x_k = x_k' r_k \\ 0 & \text{else.} \end{array} \right. \tag{41}$$

Inserting this into (11) with $m_k = m$ one can show that $\dot{y}_k^{AI} = \dot{y}_k^{FM}$, where $\dot{y}_k^{FM}$ has been defined in (39). The proof is very simple since the FM model has already a rather general structure, which is similar to the full AI model.

We expect no problem in going from FMξ to AIξ. The only thing the AIξ model has to learn, is to ignore the x' feedbacks as all information is already contained in r. This task is simple as every cycle provides one data point for a simple function to learn.

Remark. TSP seems to be trivial in the AIμ model but non-trivial in the AIξ model. The reason being that (39) just implements an internal complete search as $\mu(f) = \delta_{ff^{TSP}}$

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[14]See [Sol99] for a discussion on symmetric universal distributions on unordered data.

contains all necessary information. AIμ outputs from the very beginning, the exact minimum of f^{TSP}. This "solution" is, of course, unacceptable from performance perspective. As long as we give no efficient approximation ξ^c of ξ, we have not contributed anything to a solution of the TSP by using AIξ^c. The same is true for any other problem where f is computable and easily accessible. Therefore, TSP is not (yet) a good example because all we have done is to replace an NP complete problem with the uncomputable AIξ model or by a computable AIξ^c model, for which we have said nothing about computation time yet. It is simply an overkill to reduce simple problems to AIξ. TSP is a simple problem in this respect, until we consider the AIξ^c model seriously. For the other examples, where f is inaccessible or complicated, an AIξ^c model would provide a true solution to the minimization problem as an explicit definition of f is not needed for AIξ and AIξ^c. A computable version of AIξ will be defined in Section 6.

5.4   Supervised Learning from Examples (EX)

The developed AI models provide a frame for reinforcement learning. The environment provides feedback r, informing the agent about the quality of its last (or earlier) output y; it assigns reward r to output y. In this sense, reinforcement learning is explicitly integrated into the AIμ/ξ models. AIμ maximizes the true expected reward, whereas the AIξ model is a universal, environment independent, reinforcement learning algorithm.

There is another type of learning method: Supervised learning by presentation of examples (EX). Many problems learned by this method are association problems of the following type. Given some examples $x \in R \subset \mathcal{X}$, the agent should reconstruct, from a partially given x', the missing or corrupted parts, i.e. complete x' to x such that relation R contains x. In many cases, $\mathcal{X}$ consists of pairs (z,v), where v is the possibly missing part.

Applications/examples. Learning functions by presenting $(z,f(z))$ pairs and asking for the function value of z by presenting $(z,?)$ falls into this category.

A basic example is learning properties of geometrical objects coded in some way. E.g. if there are 18 different objects characterized by their size (small or big), their colors (red, green or blue) and their shapes (square, triange, circle), then $(object,property) \in R$ if the *object* possesses the *property*. Here, R is a relation which is not the graph of a single valued function.

When teaching a child, by pointing to objects and saying "this is a tree" or "look how green" or "how beautiful", one establishes a relation of $(object,property)$ pairs in R. Pointing to a (possibly different) tree later and asking "what is this ?" corresponds to a partially given pair $(object,?)$, where the missing part "?" should be completed by the child saying "tree".

A final example we want to give is chess. We have seen that, in principle, chess can be learned by reinforcement learning. In the extreme case the environment only provides reward $r = 1$ when the agent wins. The learning rate is completely inacceptable from a practical point of view. The reason is the very low amount of information feedback. A more practical method of teaching chess is to present example games in the form of

sensible (*board-state,move*) sequences. They contain information about legal and good moves (but without any explanation). After several games have been presented, the teacher could ask the agent to make its own move by presenting (*board-state*,?) and then evaluate the answer of the agent.

Supervised learning with the AIμ/ξ model. Let us define the EX model as follows: The environment presents inputs $x'_{k-1} = z_k v_k \equiv (z_k, v_k) \in R \cup (Z \times \{?\}) \subset Z \times (\mathcal{Y} \cup \{?\}) = \mathcal{X}'$ to the agent in cycle $k-1$. The agent is expected to output y_k in the next cycle, which is evaluated with $r_k = 1$ if $(z_k, y_k) \in R$ and 0 otherwise. To simplify the discussion, an output y_k is expected and evaluated even when $v_k (\neq ?)$ is given. To complete the description of the environment, the probability distribution $\mu_R(\underline{x'_1...x'_n})$ of the examples and questions x'_i (depending on R) has to be given. Wrong examples should not occur, i.e. μ_R should be 0 if $x'_i \notin R \cup (Z \times \{?\})$ for some $1 \leq i \leq n$. The relations R might also be probability distributed with $\sigma(\underline{R})$. The example prior probability in this case is

$$\mu(\underline{x'_1...x'_n}) \;=\; \sum_R \mu_R(\underline{x'_1...x'_n}) \cdot \sigma(\underline{R}) \tag{42}$$

The knowledge of the valuation r_k on output y_k restricts the possible relations R, consistent with $R(z_k, y_k) = r_k$, where $R(z, y) := 1$ if $(z, y) \in R$ and 0 otherwise. The prior probability for the input sequence $x_1...x_n$ if the output sequence is $y_1...y_n$, is therefore

$$\mu^{AI}(y_1\underline{x}_1...y_n\underline{x}_n) \;=\; \sum_{R:\forall 1 < i \leq n[R(z_i, y_i) = r_i]} \mu_R(\underline{x'_1...x'_n}) \cdot \sigma(\underline{R})$$

where $x_i = r_i x'_i$ and $x'_{i-1} = z_i v_i$ with $v_i \in \mathcal{Y} \cup \{?\}$. In the I/O sequence $y_1 x_1 y_2 x_2... = y_1 r_1 z_2 v_2 y_2 r_2 z_3 v_3...$ the $r_1 y_1$ are dummies, after which regular behavior starts with example (z_2, v_2).

The AIμ model is optimal by construction of μ^{AI}. For computable prior μ_R and σ, we expect a near optimal behavior of the universal AIξ model if μ_R additionally satisfies some separability property. In the following, we give some motivation why the AIξ model takes into account the supervisor information contained in the examples and why it learns faster than by reinforcement.

We keep R fixed and assume $\mu_R(x'_1...x'_n) = \mu_R(x'_1) \cdot ... \cdot \mu_R(x'_n) \neq 0 \Leftrightarrow x'_i \in R \cup (Z \times \{?\}) \; \forall i$ to simplify the discussion. Short codes q contribute most to $\xi^{AI}(y_1\underline{x}_1...y_n\underline{x}_n)$. As $x'_1...x'_n$ is distributed according to the computable probability distribution μ_R, a short code of $x'_1...x'_n$ for large enough n is a Huffman code w.r.t. the distribution μ_R. So we expect μ_R and hence R to be coded in the dominant contributions to ξ^{AI} in some way, where the plausible assumption was made that the y on the input tape do not matter. Much more than one bit per cycle will usually be learned, i.e. relation R will be learned in $n \ll K(R)$ cycles by appropriate examples. This coding of R in q evolves independently of the feedbacks r. To maximize the feedback r_k, the agent has to learn to output a y_k with $(z_k, y_k) \in R$. The agent has to invent a program extension q' to q, which extracts z_k from $x_{k-1} = (z_k, ?)$ and searches for and outputs a y_k with $(z_k, y_k) \in R$. As R is already coded in q, q' can re-use this coding of R in q. The size of the extension q' is, therefore, of order

1. To learn this q', the agent requires feedback r with information content $O(1) = K(q')$ only.

Let us compare this with reinforcement learning, where only $x'_k = (z_k, ?)$ pairs are presented. A coding of R in a short code q for $x'_1...x'_n$ is of no use and will therefore be absent. Only the rewards r force the agent to learn R. q' is therefore expected to be of size $K(R)$. The information content in the $r's$ must be of the order $K(R)$. In practice, there are often only very few $r_k = 1$ at the beginning of the learning phase and the information content in $r_1...r_n$ is much less than n bits. The required number of cycles to learn R by reinforcement is, therefore, at least but in many cases much larger than $K(R)$.

Although AIξ was never designed or told to learn supervised, it learns how to take advantage of the examples from the supervisor. μ_R and R are learned from the examples, the rewards r are not necessary for this process. The remaining task of learning how to learn supervised is then a simple task of complexity $O(1)$, for which the rewards r are necessary.

5.5 Other Aspects of Intelligence

In AI, a variety of general ideas and methods have been developed. In the previous subsections, we have seen how several problem classes can be formulated within AIξ. As we claim universality of the AIξ model, we want to enlight which of, and how the other AI methods are incorporated in the AIξ model, by looking at its structure. Some methods are directly included, others are or should be emergent. We do not claim the following list to be complete.

Probability theory and *utility theory* are the heart of the AIμ/ξ models. The probabilities are the true/universal behaviors of the environment. The utility function is what we called total reward, which should be maximized. Maximization of an expected utility function in a probabilistic environment is usually called *sequential decision theory*, and is explicitly integrated in full generality in our model. In a sense this includes probabilistic (a generalization of deterministic) *reasoning*, where the object of reasoning are not true and false statements, but the prediction of the environmental behavior. *Reinforcement Learning* is explicitly built in, due to the rewards. Supervised learning is an emergent phenomenon (Section 5.4). *Algorithmic information theory* leads us to use ξ as a universal estimate for the prior probability μ.

For horizon > 1, the expectimax series in (11) and the process of selecting maximal values may be interpreted as abstract *planning*. The expectimax series is a form of *informed search*, in the case of AIμ, and *heuristic search*, for AIξ, where ξ could be interpreted as a heuristic for μ. The minimax strategy of *game playing* in case of AIμ is also subsumed. The AIξ model converges to the minimax strategy if the environment is a minimax player but it can also take advantage of environmental players with limited rationality. *Problem solving* occurs (only) in the form of how to maximize the expected future reward.

Knowledge is accumulated by AIξ and is stored in some form not specified further on the working tape. Any kind of information in any representation on the inputs y is

exploited. The problem of *knowledge engineering* and *representation* appears in the form of how to train the AIξ model. More practical aspects, like *language or image processing* have to be learned by AIξ from scratch.

Other theories, like *fuzzy logic*, *possibility theory*, *Dempster-Shafer theory*, ... are partly outdated and partly reducible to Bayesian probability theory [Che85, Che88]. The interpretation and consequences of the evidence gap $g := 1 - \sum_{x_k} \xi(yx_{<k}y\underline{x}_k) > 0$ in ξ may be similar to those in Dempster-Shafer theory. Boolean logical reasoning about the external world plays, at best, an emergent role in the AIξ model.

Other methods, which don't seem to be contained in the AIξ model might also be emergent phenomena. The AIξ model has to construct short codes of the environmental behavior, the AI$\xi^{\tilde{t}\tilde{l}}$ (see next section) has to construct short action programs. If we would analyze and interpret these programs for realistic environments, we might find some of the unmentioned or unused or new AI methods at work in these programs. This is, however, pure speculation at this point. More important: when trying to make AIξ practically usable, some other AI methods, like genetic algorithms or neural nets, may be useful.

The main thing we wanted to point out is that the AIξ model does not lack any important known property of intelligence or known AI methodology. What *is* missing, however, are computational aspects, which are addressed in the next section.

6   Time Bounded AIXI Model

Until now, we have not bothered with the non-computability of the universal probability distribution ξ. As all universal models in this paper are based on ξ, they are not effective in this form. In this section, we outline how the previous models and results can be modified/generalized to the time-bounded case. Indeed, the situation is not as bad as it could be. ξ is enumerable and $\dot{y}_k$ is still approximable, i.e. there exists an algorithm that will produce a sequence of outputs eventually converging to the exact output $\dot{y}_k$, but we can never be sure whether we have already reached it. Besides this, the convergence is extremely slow, so this type of asymptotic computability is of no direct (practical) use, but will nevertheless, be important later.

Let $\tilde{p}$ be a program which calculates within a reasonable time $\tilde{t}$ per cycle, a reasonable intelligent output, i.e. $\tilde{p}(\dot{x}_{<k}) = \dot{y}_{1:k}$. This sort of computability assumption, that a general purpose computer of sufficient power is able to behave in an intelligent way, is the very basis of AI, justifying the hope to be able to construct agents which eventually reach and outperform human intelligence. For a contrary viewpoint see [Pen89, Pen94]. It is not necessary to discuss here, what is meant by 'reasonable time/intelligence' and 'sufficient power'. What we are interested in, in this section, is whether there is a computable version AI$\xi^{\tilde{t}}$ of the AIξ agent which is superior or equal to any p with computation time per cycle of at most $\tilde{t}$. By 'superior', we mean 'more intelligent', so what we need is an order relation (like) (10) for intelligence.

The best result we could think of would be an AI$\xi^{\tilde{t}}$ with computation time $\le \tilde{t}$ at least as intelligent as any p with computation time $\le \tilde{t}$. If AI is possible at all, we would have

reached the final goal, the construction of the most intelligent algorithm with computation time $\leq \tilde{t}$. Just as there is no universal measure in the set of computable measures (within time $\tilde{t}$), such an $\text{AI}\xi^{\tilde{t}}$ may neither exist.

What we can realistically hope to construct, is an $\text{AI}\xi^{\tilde{t}}$ agent of computation time $c \cdot \tilde{t}$ per cycle for some constant c. The idea is to run all programs p of length $\leq \tilde{l} := l(\tilde{p})$ and time $\leq \tilde{t}$ per cycle and pick the best output. The total computation time is $c \cdot \tilde{t}$ with $c = 2^{\tilde{l}}$. This sort of idea of 'typing monkeys' with one of them eventually writing Shakespeare, has been applied in various forms and contexts in theoretical computer science. The realization of this *best vote* idea, in our case, is not straightforward and will be outlined in this section. An idea related to this, is that of basing the decision on the majority of algorithms. This 'democratic vote' idea has been used in [LW94, Vov92] for sequence prediction, and is referred to as 'weighted majority' there.

6.1   Time Limited Probability Distributions

In the literature one can find time limited versions of Kolmogorov complexity [Dal73, Dal77, Ko86] and the time limited universal semimeasure [LV91, LV97, Sch02b]. In the following, we utilize and adapt the latter and see how far we get. One way to define a time-limited universal chronological semimeasure is as a sum over all enumerable chronological semimeasures with computation time at most $\tilde{t}$ and of size at most $\tilde{l}$.

$$\xi^{\tilde{t}\tilde{l}}(y\underline{x}_{1:n}) \; := \sum_{\rho \, : \, l(\rho) \leq \tilde{l} \, \wedge \, t(\rho) \leq \tilde{t}} 2^{-l(\rho)}\rho(y\underline{x}_{1:n}) \tag{43}$$

One can show that $\xi^{\tilde{t}\tilde{l}}$ reduces to ξ^{AI} defined in (22) for $\tilde{t}, \tilde{l} \to \infty$. Let us assume that the true environmental prior probability μ^{AI} is equal to or sufficiently accurately approximated by a ρ with $l(\rho) \leq \tilde{l}$ and $t(\rho) \leq \tilde{t}$ with $\tilde{t}$ and $\tilde{l}$ of reasonable size. There are several AI problems that fall into this class. In function minimization of Section 5.3, the computation of f and μ^{FM} are often feasible. In many cases, the sequences of Section 5.1 which should be predicted, can be easily calculated when μ^{SP} is known. In a classifier problem, the probability distribution μ^{CF}, according to which examples are presented, is, in many cases, also elementary. But not all AI problems are of this 'easy' type. For the strategic games of Section 5.2, the environment is usually, itself, a highly complex strategic player with a μ^{SG} that is difficult to calculate, although one might argue that the environmental player may have limited capabilities too. But it is easy to think of a difficult to calculate physical (probabilistic) environment like the chemistry of biomolecules.

The number of interesting applications makes this restricted class of AI problems, with time and space bounded environment $\mu^{\tilde{t}\tilde{l}}$, worth being studied. Superscripts to a probability distribution except for $\xi^{\tilde{t}\tilde{l}}$ indicate their length and maximal computation time. $\xi^{\tilde{t}\tilde{l}}$ defined in (43), with a yet to be determined computation time, multiplicatively dominates all $\mu^{\tilde{t}\tilde{l}}$ of this type. Hence, an $\text{AI}\xi^{\tilde{t}\tilde{l}}$ model, where we use $\xi^{\tilde{t}\tilde{l}}$ as prior probability, is universal, relative to all $\text{AI}\mu^{\tilde{t}\tilde{l}}$ models in the same way as $\text{AI}\xi$ is universal to $\text{AI}\mu$ for all enumerable chronological semimeasures μ. The argmax_{y_k} in (23) selects a y_k for which

$\xi^{\tilde{t}\tilde{l}}$ has the highest expected utility V_{km_k}, where $\xi^{\tilde{t}\tilde{l}}$ is the weighted average over the $\rho^{\tilde{t}\tilde{l}}$. $\dot{y}_k^{AI\xi^{\tilde{t}\tilde{l}}}$ is determined by a weighted majority. We expect $AI\xi^{\tilde{t}\tilde{l}}$ to outperform all (bounded) $AI\rho^{\tilde{t}\tilde{l}}$, analogous to the unrestricted case.

In the following we analyze the computability properties of $\xi^{\tilde{t}\tilde{l}}$ and $AI\xi^{\tilde{t}\tilde{l}}$, i.e. of $\dot{y}_k^{AI\xi^{\tilde{t}\tilde{l}}}$. To compute $\xi^{\tilde{t}\tilde{l}}$ according to the definition (43) we have to enumerate all chronological enumerable semimeasures $\rho^{\tilde{t}\tilde{l}}$ of length $\leq \tilde{l}$ and computation time $\leq \tilde{t}$. This can be done similarly to the unbounded case as described in [LV97, Hut00]. All $2^{\tilde{l}}$ enumerable functions of length $\leq \tilde{l}$, computable within time $\tilde{t}$ have to be converted to chronological probability distributions. For this, one has to evaluate each function for $|\mathcal{X}| \cdot k$ different arguments. Hence, $\xi^{\tilde{t}\tilde{l}}$ is computable within time[15] $t(\xi^{\tilde{t}\tilde{l}}(y\underline{x}_{1:k})) = O(|\mathcal{X}| \cdot k \cdot 2^{\tilde{l}} \cdot \tilde{t})$. The computation time of $\dot{y}_k^{AI\xi^{\tilde{t}\tilde{l}}}$ depends on the size of $\mathcal{X}$, $\mathcal{Y}$ and m_k. $\xi^{\tilde{t}\tilde{l}}$ has to be evaluated $|\mathcal{Y}|^{h_k}|\mathcal{X}|^{h_k}$ times in (23). It is possible to optimize the algorithm and perform the computation within time

$$t(\dot{y}_k^{AI\xi^{\tilde{t}\tilde{l}}}) \;=\; O(|\mathcal{Y}|^{h_k}|\mathcal{X}|^{h_k} \cdot 2^{\tilde{l}} \cdot \tilde{t}) \tag{44}$$

per cycle. If we assume that the computation time of $\mu^{\tilde{t}\tilde{l}}$ is exactly $\tilde{t}$ for all arguments, the brute force time $\bar{t}$ for calculating the sums and maxs in (12) is $\bar{t}(\dot{y}_k^{AI\mu^{\tilde{t}\tilde{l}}}) \geq |\mathcal{Y}|^{h_k}|\mathcal{X}|^{h_k} \cdot \tilde{t}$. Combining this with (44), we get

$$t(\dot{y}_k^{AI\xi^{\tilde{t}\tilde{l}}}) \;=\; O(2^{\tilde{l}} \cdot \bar{t}(\dot{y}_k^{AI\mu^{\tilde{t}\tilde{l}}}))$$

This result has the proposed structure, that there is a universal $AI\xi^{\tilde{t}\tilde{l}}$ agent with computation time $2^{\tilde{l}}$ times the computation time of a special $AI\mu^{\tilde{t}\tilde{l}}$ agent.

Unfortunately, the class of $AI\mu^{\tilde{t}\tilde{l}}$ systems with brute force evaluation of $\dot{y}_k$, according to (12) is completely uninteresting from a practical point of view. E.g. in the context of chess, the above result says that the $AI\xi^{\tilde{t}\tilde{l}}$ is superior within time $2^{\tilde{l}} \cdot \tilde{t}$ to any brute force minimax strategy of computation time $\tilde{t}$. Even if the factor of $2^{\tilde{l}}$ in computation time would not matter, the $AI\xi^{\tilde{t}\tilde{l}}$ agent is, nevertheless practically useless, as a brute force minimax chess player with reasonable time $\tilde{t}$ is a very poor player.

Note, that in the case of binary sequence prediction ($h_k=1$, $|\mathcal{Y}|=|\mathcal{X}|=2$) the computation time of ρ coincides with that of $\dot{y}_k^{AI\rho}$ within a factor of 2. The class $AI\rho^{\tilde{t}\tilde{l}}$ includes *all* non-incremental sequence prediction algorithms of size $\leq \tilde{l}$ and computation time $\leq \tilde{t}/2$. By non-incremental, we mean that no information of previous cycles is taken into account for speeding up the computation of $\dot{y}_k$ of the current cycle.

The shortcomings (mentioned and unmentioned ones) of this approach are cured in the next subsection, by deviating from the standard way of defining a time bounded ξ as a sum over functions or programs.

6.2 The Idea of the Best Vote Algorithm

A general agent is a chronological program $p(x_{<k})=y_{1:k}$. This form, introduced in Section 2.4, is general enough to include any AI system (and also less intelligent systems). In the

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[15]We assume that a TM can be simulated by another in linear time.

following, we are interested in programs p of length $\leq \tilde{l}$ and computation time $\leq \tilde{t}$ per cycle. One important point in the time-limited setting is that p should be incremental, i.e. when computing y_k in cycle k, the information of the previous cycles stored on the working tape can be re-used. Indeed, there is probably no practically interesting, non-incremental AI system at all.

In the following, we construct a policy p^*, or more precisely, policies p_k^* for every cycle k that outperform all time and length limited AI systems p. In cycle k, p_k^* runs all $2^{\tilde{l}}$ programs p and selects the one with the best output y_k. This is a 'best vote' type of algorithm, as compared to the 'weighted majority' like algorithm of the last subsection. The ideal measure for the quality of the output would be the ξ-expected future reward

$$V_{km}^{p\xi}(\dot{y}\ddot{x}_{<k}) \; := \; \sum_{q \in \dot{Q}_k} 2^{-l(q)} V_{km}^{pq} \quad , \quad V_{km}^{pq} \; := \; r(x_k^{pq}) + ... + r(x_m^{pq}) \tag{45}$$

The program p which maximizes $V_{km_k}^{p\xi}$ should be selected. We have dropped the normalization $\mathcal{N}$ unlike in (25), as it is independent of p and does not change the order relation which we are solely interested in here. Furthermore, without normalization, $V_{km}^{*\xi}(\dot{y}\ddot{x}_{<k}) := \max_{p \in \dot{P}} V_{km}^{p\xi}(\dot{y}\ddot{x}_{<k})$ is enumerable, which will be important later.

6.3 Extended Chronological Programs

In the (functional form of the) AIξ model it was convenient to maximize V_{km_k} over all $p \in \dot{P}_k$, i.e. all p consistent with the current history $\dot{y}\ddot{x}_{<k}$. This was no restriction, because for every possibly inconsistent program p there exists a program $p' \in \dot{P}_k$ consistent with the current history and identical to p for all future cycles $\geq k$. For the time limited best vote algorithm p^* it would be too restrictive to demand $p \in \dot{P}_k$. To prove universality, one has to compare *all* $2^{\tilde{l}}$ algorithms in every cycle, not just the consistent ones. An inconsistent algorithm may become the best one in later cycles. For inconsistent programs we have to include the $\dot{y}_k$ into the input, i.e. $p(\dot{y}\ddot{x}_{<k}) = y_{1:k}^p$ with $\dot{y}_i \neq y_i^p$ possible. For $p \in \dot{P}_k$ this was not necessary, as p knows the output $\dot{y}_k \equiv y_k^p$ in this case. The r_i^{pq} in the definition of V_{km} are the valuations emerging in the I/O sequence, starting with $\dot{y}\ddot{x}_{<k}$ (emerging from p^*) and then continued by applying p and q with $\dot{y}_i := y_i^p$ for $i \geq k$.

Another problem is that we need V_{km_k} to select the best policy, but unfortunately V_{km_k} is uncomputable. Indeed, the structure of the definition of V_{km_k} is very similar to that of $\dot{y}_k$, hence a brute force approach to approximate V_{km_k} requires too much computation time as for $\dot{y}_k$. We solve this problem in a similar way, by supplementing each p with a program that estimates V_{km_k} by w_k^p within time $\tilde{t}$. We combine the calculation of y_k^p and w_k^p and extend the notion of a chronological program once again to

$$p(\dot{y}\ddot{x}_{<k}) \; = \; w_1^p y_1^p ... w_k^p y_k^p \tag{46}$$

with chronological order $w_1^p y_1^p \dot{y}_1 \dot{x}_1 w_2^p y_2^p \dot{y}_2 \dot{x}_2 ....$

6.4   Valid Approximations

p might suggest any output y_k^p but it is not allowed to rate it with an arbitrarily high w_k^p if we want w_k^p to be a reliable criterion for selecting the best p. We demand that no policy is allowed to claim that it is better than it actually is. We define a (logical) predicate VA(p) called *valid approximation*, which is true if, and only if, p always satisfies $w_k^p \le V_{km_k}^{p\xi}$, i.e. never overrates itself.

$$\text{VA}(p) \;\equiv\; \forall k \forall w_1^p y_1^p \dot{y}_1 \dot{x}_1 ... w_k^p y_k^p : p(\ddot{y}\ddot{x}_{<k}) = w_1^p y_1^p ... w_k^p y_k^p \Rightarrow w_k^p \le V_{km_k}^{p\xi}(\ddot{y}\ddot{x}_{<k}) \qquad (47)$$

In the following, we restrict our attention to programs p, for which VA(p) can be proven in some formal axiomatic agent. A very important point is that $V_{km_k}^{*\xi}$ is enumerable. This ensures the existence of sequences of programs $p_1, p_2, p_3,...$ for which VA(p_i) can be proven and $\lim_{i\to\infty} w_k^{p_i} = V_{km_k}^{*\xi}$ for all k and all I/O sequences. p_i may be defined as the naive (non-halting) approximation scheme (by enumeration) of $V_{km_k}^{*\xi}$ terminated after i time steps and using the approximation obtained so far for $w_k^{p_i}$ together with the corresponding output $y_k^{p_i}$. The convergence $w_k^{p_i} \overset{i\to\infty}{\longrightarrow} V_{km_k}^{*\xi}$ ensures that $V_{km_k}^{*\xi}$, which we claimed to be the universally optimal value, can be approximated by p with provable $VA(p)$ arbitrarily well, when given enough time. The approximation is not uniform in k, but this does not matter as the selected p is allowed to change from cycle to cycle.

Another possibility would be to consider only those p which check $w_k^p \le V_{km_k}^{p\xi}$ online in every cycle, instead of the pre-check VA(p), either by constructing a proof (on the working tape) for this special case, or $w_k^p \le V_{km_k}^{p\xi}$ is already evident by the construction of w_k^p. In cases where p cannot guarantee $w_k^p \le V_{km_k}^{p\xi}$ it sets $w_k = 0$ and, hence, trivially satisfies $w_k^p \le V_{km_k}^{p\xi}$. On the other hand, for these p it is also no problem to prove VA(p) as one has simply to analyze the internal structure of p and recognize that p shows the validity internally itself, cycle by cycle, which is easy by assumption on p. The cycle by cycle check is, therefore, a special case of the pre-proof of VA(p).

6.5   Effective Intelligence Order Relation

In Section 4.1 we have introduced an intelligence order relation $\succeq$ on AI systems, based on the expected reward $V_{km_k}^{p\xi}$. In the following we need an order relation $\succeq^c$ based on the claimed reward w_k^p which might be interpreted as an approximation to $\succeq$.

Definition 13 (Effective intelligence order relation) *We call p effectively more or equally intelligent than p' if*

$$p \succeq^c p' \; :\Leftrightarrow \; \forall k \forall \ddot{y}\ddot{x}_{<k} \exists w_{1:n} w'_{1:n} :$$
$$p(\ddot{y}\ddot{x}_{<k}) = w_1 * ... w_k * \;\land\; p'(\ddot{y}\ddot{x}_{<k}) = w'_1 * ... w'_k * \;\land\; w_k \ge w'_k$$

i.e. if p always claims higher reward estimate w than p'.

$\succeq^c$ is a co-enumerable partial order relation on extended chronological programs. Restricted to valid approximations it orders the policies w.r.t. the quality of their outputs *and* their ability to justify their outputs with high w_k.

6.6   The Universal Time Bounded AIXI$\widetilde{tl}$ Agent

In the following, we describe the algorithm p^* underlying the universal time bounded AI$\xi^{\widetilde{tl}}$ agent. It is essentially based on the selection of the best algorithms p_k^* out of the time $\tilde{t}$ and length $\tilde{l}$ bounded p, for which there exists a proof of VA(p) with length $\leq l_P$.

1. Create all binary strings of length l_P and interpret each as a coding of a mathematical proof in the same formal logic agent in which VA($\cdot$) has been formulated. Take those strings which are proofs of VA(p) for some p and keep the corresponding programs p.
2. Eliminate all p of length $> \tilde{l}$.
3. Modify all p in the following way: all output $w_k^p y_k^p$ is temporarily written on an auxiliary tape. If p stops in $\tilde{t}$ steps the internal 'output' is copied to the output tape. If p does not stop after $\tilde{t}$ steps a stop is forced and $w_k = 0$ and some arbitrary y_k is written on the output tape. Let P be the set of all those modified programs.
4. Start first cycle: $k := 1$.
5. Run every $p \in P$ on extended input $\ddot{y}\ddot{x}_{<k}$, where all outputs are redirected to some auxiliary tape: $p(\ddot{y}\ddot{x}_{<k}) = w_1^p y_1^p ... w_k^p y_k^p$. This step is performed incrementally by adding yx_{k-1} for $k > 1$ to the input tape and continuing the computation of the previous cycle.
6. Select the program p with highest claimed reward w_k^p: $p_k^* := \operatorname{argmax}_p w_k^p$.
7. Write $\dot{y}_k := y_k^{p_k^*}$ to the output tape.
8. Receive input $\dot{x}_k$ from the environment.
9. Begin next cycle: $k := k+1$, goto step 5.

It is easy to see that the following theorem holds.

Theorem 14 (Optimality of AIXItl) *Let p be any extended chronological (incremental) program like (46) of length $l(p) \leq \tilde{l}$ and computation time per cycle $t(p) \leq \tilde{t}$, for which there exists a proof of VA(p) defined in (47) of length $\leq l_P$. The algorithm p^* constructed in the last subsection, depending on $\tilde{l}$, $\tilde{t}$ and l_P but not on p, is effectively more or equally intelligent, according to $\succeq^c$ (see Definition 13) than any such p. The size of p^* is $l(p^*) = O(\log(\tilde{l} \cdot \tilde{t} \cdot l_P))$, the setup-time is $t_{setup}(p^*) = O(l_P \cdot 2^{l_P})$ and the computation time per cycle is $t_{cycle}(p^*) = O(2^{\tilde{l}} \cdot \tilde{t})$.*

Roughly speaking, the theorem says, that if there exists a computable solution to some (or all) AI problem(s) at all, the explicitly constructed algorithm p^* is such a solution. Although this theorem is quite general, there are some limitations and open questions which we discuss in the following.

The construction of the algorithm p^* needs the specification of a formal logic system $(\forall, \lambda, y_i, c_i, f_i, R_i, \rightarrow, \wedge, =, ...)$, and axioms, and inference rules. A proof is a sequence of formulas, where each formula is either an axiom or inferred from previous formulas in the sequence by applying the inference rules. Details can be found in [Hut02a] in a related construction or in any textbook on logic or proof theory, e.g. [Fit96, Sho67]. We only

need to know that *provability* and *Turing Machines* can be formalized. The setup time in the main theorem is just the time needed to verify the 2^{l_P} proofs, each needing time $O(l_P^2)$.

6.7 Limitations and Open Questions

- Formally, the total computation time of p^* for cycles $1...k$ increases linearly with k, i.e. is of order $O(k)$ with a coefficient $2^{\tilde{l}} \cdot \tilde{t}$. The unreasonably large factor $2^{\tilde{l}}$ is a well known drawback in best/democratic vote models and will be taken without further comments, whereas the factor $\tilde{t}$ can be assumed to be of reasonable size. If we don't take the limit $k \to \infty$ but consider reasonable k, the practical usefulness of the time bound on p^* is somewhat limited, due to the additional additive constant $O(l_P \cdot 2^{l_P})$. It is much larger than $k \cdot 2^{\tilde{l}} \cdot \tilde{t}$ as typically $l_P \gg l(\text{VA}(p)) \geq l(p) \equiv \tilde{l}$.

- p^* is superior only to those p which justify their outputs (by large w_k^p). It might be possible that there are p which produce good outputs y_k^p within reasonable time, but it takes an unreasonably long time to justify their outputs by sufficiently high w_k^p. We do not think that (from a certain complexity level onwards) there are policies where the process of constructing a good output is completely separated from some sort of justification process. But this justification might not be translatable (at least within reasonable time) into a reasonable estimate of $V_{km_k}^{p\xi}$.

- The (inconsistent) programs p must be able to continue strategies started by other policies. It might happen that a policy p steers the environment to a direction for which p is specialized. A 'foreign' policy might be able to displace p only between loosely connected episodes. There is probably no problem for factorizable μ. Think of a chess game, where it is usually very difficult to continue the game/strategy of a different player. When the game is over, it is usually advantageous to replace a player by a better one for the next game. There might also be no problem for sufficiently separable μ.

- There might be (efficient) valid approximations p for which $\text{VA}(p)$ is true but not provable, or for which only a very long $(> l_P)$ proof exists.

6.8 Remarks

- The idea of suggesting outputs and justifying them by proving reward bounds implements one aspect of human thinking. There are several possible reactions to an input. Each reaction possibly has far reaching consequences. Within a limited time one tries to estimate the consequences as well as possible. Finally, each reaction is valued and the best one is selected. What is inferior to human thinking is, that the estimates w_k^p must be rigorously proved and the proofs are constructed by blind exhaustive search, further, that *all* behaviors p of length $\leq \tilde{l}$ are checked. It is inferior 'only' in the sense of necessary computation time but not in the sense of the quality of the outputs.

- In practical applications there are often cases with short and slow programs p_s performing some task T, e.g. the computation of the digits of π, for which there exist long but quick programs p_l too. If it is not too difficult to prove that this long program is equivalent to the short one, then it is possible to prove $K^{t(p_l)}(T) \stackrel{+}{\leq} l(p_s)$ with K^t being the time bounded Kolmogorov complexity. Similarly, the method of proving bounds w_k for V_{km_k} can give high lower bounds without explicitly executing these short and slow programs, which mainly contribute to V_{km_k}.

- Dovetailing all length and time-limited programs is a well known elementary idea (typing monkeys). The crucial part which has been developed here, is the selection criterion for the most intelligent agent.

- By construction of AI$\xi^{\tilde{t}\tilde{l}}$ and due to the enumerability of V_{km_k}, ensuring arbitrary close approximations of V_{km_k} we expect that the behavior of AI$\xi^{\tilde{t}\tilde{l}}$ converges to the behavior of AIξ in the limit $\tilde{t}, \tilde{l}, l_P \to \infty$ in a sense.

- Depending on what you know/assume that a program p of size $\tilde{l}$ and computation time per cycle $\tilde{t}$ is able to achieve, the computable AI$\xi^{\tilde{t}\tilde{l}}$ model will have the same capabilities. For the strongest assumption of the existence of a Turing machine, which outperforms human intelligence, the AI$\xi^{\tilde{t}\tilde{l}}$ will do too, within the same time frame up to a (unfortunately very large) constant factor.

7   Discussion

This section reviews what has been achieved in the article and discusses some otherwise unmentioned topics of general interest. We remark on various topics, including concurrent actions and observations, the choice of the I/O spaces, treatment of encrypted information, and peculiarities of mortal embodies agents. We continue with an outlook on further research. Since many ideas have already been presented in the various sections, we concentrate on non-technical open questions of general importance, including optimality, down-scaling, implementation, approximation, elegance, extra knowledge, and training of/for AIXI(tl). We also include some (personal) remarks on non-computable physics, the number of wisdom Ω, and consciousness. As it should be, the article concludes with conclusions.

7.1   General Remarks

This section remarks on some otherwise unmentioned topics of general interest. The logically disconnected subsections discuss concurrent actions and observations, the choice of the I/O spaces, treatment of encrypted information, and peculiarities of mortal embodies agents.

Game theory. In game theory [OR94] one often wants to model the situation of simultaneous actions, whereas the AIξ models has serial I/O. Simultaneity can be simulated by withholding the environment from the current agent's output y_k, until x_k has been

received by the agent. Formally, this means that $\mu(y\!x_{<k}y\underline{x}_k)$ is independent of the last output y_k. The AIξ agent is already of simultaneous type in an abstract view if the behavior p is interpreted as the action. In this sense, AIXI is the action p^* which maximizes the utility function (reward), under the assumption that the environment acts according to ξ. The situation is different from game theory as the environment ξ is not a second 'player' that tries to optimize his own utility (see Section 5.2).

Input/output spaces. In various examples we have chosen differently specialized input and output spaces $\mathcal{X}$ and $\mathcal{Y}$. It should be clear that, in principle, this is unnecessary, as large enough spaces $\mathcal{X}$ and $\mathcal{Y}$ (e.g. the set of strings of length 2^{32}) serve every need and can always be Turing-reduced to the specific presentation needed internally by the AIXI agent itself. But it is clear that, using a generic interface, such as camera and monitor for learning tic-tac-toe, for example, adds the task of learning vision and drawing.

How AIXI(tl) deals with encrypted information. If you encrypt a message with a key of size k but then you also give away the key, then the overall algorithm A for encryption has size $O(1)$. $A =$ "Take key and message and encrypt message with key". The system only has to "learn" A^{-1} which can itself be described in length $O(1)$. Only very little information is needed to learn $O(1)$ bits. In this sense decryption is easy. The problem is that A^{-1} may be an extremely slow algorithm (e.g. finding the prime factors from the public key in RSA). But note, in AIXI we are not talking about computation time, we are only talking about information efficiency (learning in the least number of interaction cycles). This is maybe one of the key ideas to separate data efficiency from computation time efficiency. Of course in the real world computation time matters, so we invented AIXItl. AIXItl can do every job as good as the best length l and time t bounded agent, apart from time factor 2^l and a huge offset time. This offset time would be used, for instance, in the RSA example, to (once-and-for-all) find the factorization, and then, decryption is easy, of course.

Mortal embodied agents. The examples we gave in this article, particularly those in Section 5, were mainly bodiless agents: predictors, gamblers, optimizers, learners. There are some peculiarities with reinforcement learning autonomous embodied robots in real environments.

We can still reward the robot according to how well it solves the task we want it to do. A minimal requirement is that the robot's hardware functions properly. If the robot starts to malfunction its capabilities degrade, resulting in lower reward. So in an attempt to maximize reward the robot will also maintain itself. The problem is that some parts will malfunction rather quickly when no appropriate actions are performed, e.g. flat batteries, if not recharged in time. Even worse, the robot may work perfectly until the battery is nearly empty, and then suddenly stop its operation (death), resulting in (minimal) zero reward from then on. There is too little time to learn how to maintain itself before it's too late. An autonomous embodied robot cannot start from scratch but must have some rudimentary built-in capabilities (which may not be that rudimentary at all) which allows it to at least survive. Animals survive due to reflexes, innate behavior, an internal reward attached to the condition of their organs, and a guarding environment during childhood.

Different species emphasize different aspects. Reflexes and innate behaviors are stressed in lower animals versus years of safe childhood for humans. The same variety of solutions is available for constructing autonomous robots (which we will not detail here).

Another problem connected, but possibly not limited to embodied agents, especially if they are rewarded by humans, is the following: Sufficiently intelligent agents may increase their rewards by psychologically manipulating their human "teachers", or by threatening them. This is a general sociological problem which successful AI will cause, which has nothing specifically to do with AIXI. Every intelligence superior to humans is capable of manipulating the latter. In the absence of manipulable humans, e.g. where the reward structure serves a survival function, AIXI may directly hack into its reward feedback. Since this will unlikely increase its long-term survival, AIXI will probably resist this kind of manipulation (like most humans don't take hard drugs, due to their long-term catastrophic consequences).

7.2   Outlook & Open Questions

Many ideas for further studies have already been stated in the various sections of the article, especially in the Problems and Conclusions sections. This outlook only contains non-technical open questions regarding AIXI(tl) of general importance.

Value bounds. Rigorous proofs for non-asymptotic value bounds for AIξ are the major theoretical challenge – general ones, as well as tighter bounds for special environments μ, e.g. for rapidly mixing MDPs. For AIXI other performance criteria have to be found and proved. Although not necessary from a practical point of view, the study of continuous classes $\mathcal{M}$, restricted policy classes, and/or infinite $\mathcal{Y}$, $\mathcal{X}$ and m may lead to useful insights.

Scaling AIXI down. A direct implementation of the AIXItl model is, at best, possible for toy environments due to the large factor 2^l in computation time. But there are other applications of the AIXI theory. We have seen in several examples how to integrate problem classes into the AIXI model. Conversely, one can downscale the AIξ model by using more restricted forms of ξ. This could be done in the same way as the theory of universal induction has been downscaled with many insights to the Minimum Description Length principle [LV92a, Ris89] or to the domain of finite automata [FMG92]. The AIXI model might similarly serve as a super model or as the very definition of (universal unbiased) intelligence, from which specialized models could be derived.

Implementation and approximation. With a reasonable computation time, the AIXI model would be a solution of AI (see next point if you disagree). The AIXItl model was the first step, but the elimination of the factor 2^l without giving up universality will (almost certainly) be a very difficult task.[16] One could try to select programs p and prove VA(p) in a more clever way than by mere enumeration, to improve performance without destroying universality. All kinds of ideas like genetic algorithms, advanced theorem provers and

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[16]But see [Hut02a] for an elegant *theoretical* solution.

many more could be incorporated. But now we have a problem.

Computability. We seem to have transferred the AI problem just to a different level. This shift has some advantages (and also some disadvantages) but presents, in no way, a solution. Nevertheless, we want to stress that we have reduced the AI problem to (mere) computational questions. Even the most general other systems the author is aware of, depend on some (more than complexity) assumptions about the environment or it is far from clear whether they are, indeed, universally optimal. Although computational questions are themselves highly complicated, this reduction is a non-trivial result. A formal theory of something, even if not computable, is often a great step toward solving a problem and also has merits of its own, and AI should not be different in this respect (see previous item).

Elegance. Many researchers in AI believe that intelligence is something complicated and cannot be condensed into a few formulas. It is more a combining of enough *methods* and much explicit *knowledge* in the right way. From a theoretical point of view we disagree, as the AIXI model is simple and seems to serve all needs. From a practical point of view we agree to the following extent: To reduce the computational burden one should provide special purpose algorithms (*methods*) from the very beginning, probably many of them related to reduce the complexity of the input and output spaces $\mathcal{X}$ and $\mathcal{Y}$ by appropriate pre/post-processing *methods*.

Extra knowledge. There is no need to incorporate extra *knowledge* from the very beginning. It can be presented in the first few cycles in *any* format. As long as the algorithm to interpret the data is of size $O(1)$, the AIXI agent will 'understand' the data after a few cycles (see Section 5.4). If the environment μ is complicated but extra knowledge z makes $K(\mu|z)$ small, one can show that the bound (18) reduces roughly to $\ln 2 \cdot K(\mu|z)$ when $x_1 \equiv z$, i.e. when z is presented in the first cycle. The special purpose algorithms could be presented in x_1, too, but it would be cheating to say that no special purpose algorithms had been implemented in AIXI. The boundary between implementation and training is unsharp in the AIXI model.

Training. We have not said much about the training process itself, as it is not specific to the AIXI model and has been discussed in literature in various forms and disciplines [Sol86, Sch02a]. A serious discussion would be out of place. To repeat a truism, it is, of course, important to present enough knowledge x_k' and evaluate the agent output y_k with r_k in a reasonable way. To maximize the information content in the reward, one should start with simple tasks and give positive reward to approximately the better half of the outputs y_k.

7.3   The Big Questions

This subsection is devoted to the *big* questions of AI in general and the AIXI model in particular with a personal touch.

On non-computable physics & brains. There are two possible objections to AI in

general and, therefore, to AIXI in particular. Non-computable physics (which is not too weird) could make Turing computable AI impossible. As at least the world that is relevant for humans seems mainly to be computable we do not believe that it is necessary to integrate non-computable devices into an AI system. The (clever and nearly convincing) 'Gödel' argument by Penrose [Pen89, Pen94] that non-computational physics *must* exist and *is* relevant to the brain, has (in our opinion convincing) loopholes.

Evolution & the number of wisdom. A more serious problem is the evolutionary information gathering process. It has been shown that the 'number of wisdom' Ω contains a very compact tabulation of 2^n undecidable problems in its first n binary digits [Cha91]. Ω is only enumerable with computation time increasing more rapidly with n than any recursive function. The enormous computational power of evolution could have developed and coded something like Ω into our genes, which significantly guides human reasoning. In short: Intelligence could be something complicated and evolution toward it from an even cleverly designed algorithm of size $O(1)$ could be too slow. As evolution has already taken place, we could add the information from our genes or brain structure to any/our AI system, but this means that the important part is still missing and that it is principally impossible to derive an efficient algorithm from a simple formal definition of AI.

Consciousness. For what is probably the *biggest question*, that of *consciousness*, we want to give a physical analogy. Quantum (field) theory is the most accurate and universal physical theory ever invented. Although already developed in the 1930s the *big* question, regarding the interpretation of the wave function collapse, is still open. Although extremely interesting from a philosophical point of view, it is completely irrelevant from a practical point of view[17]. We believe the same to be valid for *consciousness* in the field of Artificial Intelligence. Philosophically highly interesting but practically unimportant. Whether consciousness *will* be explained some day is another question.

7.4   Conclusions

The major theme of the article was to develop a mathematical foundation of Artificial Intelligence. This is not an easy task since intelligence has many (often ill-defined) faces. More specifically, our goal was to develop a theory for rational agents acting optimally in any environment. Thereby we touched various scientific areas, including reinforcement learning, algorithmic information theory, Kolmogorov complexity, computational complexity theory, information theory and statistics, Solomonoff induction, Levin search, sequential decision theory, adaptive control theory, and many more.

We started with the observation that all tasks which require intelligence to be solved can naturally be formulated as a maximization of some expected utility in the framework of agents. We presented a functional (3) and an iterative (12) formulation of such a decision theoretic agent in Section 2, which is general enough to cover all AI problem classes, as has been demonstrated by several examples. The main remaining problem is the unknown

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[17]In the Theory of Everything, the collapse might become of 'practical' importance and must or will be solved.

prior probability distribution μ of the environment(s). Conventional learning algorithms are unsuitable, because they can neither handle large (unstructured) state spaces, nor do they converge in the theoretically minimal number of cycles, nor can they handle nonstationary environments appropriately. On the other hand, the Solomonoff's universal semimeasure ξ (17), based on ideas from algorithmic information theory, solves the problem of the unknown prior distribution for induction problems as has been demonstrated in Section 3. No explicit learning procedure is necessary, as ξ automatically converges to μ. We unified the theory of universal sequence prediction with the decision theoretic agent by replacing the unknown true prior μ by an appropriately generalized universal semimeasure ξ in Section 4. We gave various arguments that the resulting AIXI model is the most intelligent, parameter-free and environmental/application independent model possible. We defined an intelligence order relation (10) to give a rigorous meaning to this claim. Furthermore, possible solutions to the horizon problem have been discussed. In Section 5 we outlined how the AIXI model solves various problem classes. These included sequence prediction, strategic games, function minimization and, especially, learning to learn supervised. The list could easily be extended to other problem classes like classification, function inversion and many others. The major drawback of the AIXI model is that it is uncomputable, or more precisely, only asymptotically computable, which makes an implementation impossible. To overcome this problem, we constructed a modified model AIXItl, which is still effectively more intelligent than any other time t and space l bounded algorithm (Section 6). The computation time of AIXItl is of the order $t{\cdot}2^l$. A way of overcoming the large multiplicative 2^l constant has been presented in [Hut02a] at the expense of an (unfortunately even larger). Possible further research has been discussed. The main directions could be to prove general and special reward bounds, use AIXI as a super model and explore its relation to other specialized models and finally improve performance with or without giving up universality.

All in all, the results show that Artificial Intelligence can be framed by an elegant mathematical theory. Some progress has also been made towards an elegant *computational* theory of intelligence.

Annotated Bibliography

Introductory textbooks. The book of Hopcroft and Ullman, and in the new revision, co-authored by Motwani [HMU01] is a very readable elementary introduction to automata theory, formal languages, and computation theory. The Artificial Intelligence book [RN95] by Russell and Norvig gives a comprehensive overview over AI approaches in general. For an excellent introduction to Algorithmic Information Theory, Kolmogorov complexity, and Solomonoff induction one should consult the book of Li and Vitányi [LV97]. The Reinforcement Learning book by Sutton and Barto [SB98] requires no background knowledge, describes the key ideas, open problems, and great applications of this field. A tougher and more rigorous book by Bertsekas and Tsitsiklis on sequential decision theory provides all (convergence) proofs [Ber95].

Algorithmic information theory. Kolmogorov [Kol65] suggested to define the information content of an object as the length of the shortest program computing a representation of it. Solomonoff [Sol64] invented the closely related universal prior probability distribution and used it for binary sequence prediction [Sol64, Sol78] and function inversion and minimization [Sol86]. Together with Chaitin [Cha66, Cha75] this was the invention of what is now called Algorithmic Information theory. For further literature and many applications see [LV97]. Other interesting "applications" can be found in [Cha91, Sch99, VW98]. Related topics are the Weighted Majority Algorithm invented by Littlestone and Warmuth [LW94], universal forecasting by Vovk [Vov92], Levin search [Lev73], pac-learning introduced by Valiant [Val84] and Minimum Description Length [LV92a, Ris89]. Resource bounded complexity is discussed in [Dal73, Dal77, FMG92, Ko86, PF97], resource bounded universal probability in [LV91, LV97, Sch02b]. Implementations are rare and mainly due to Schmidhuber [Con97, Sch97, SZW97, Sch02a]. Excellent reviews with a philosophical touch are [LV92b, Sol97]. For an older, but general review of inductive inference see Angluin [AS83].

Sequential decision theory. The other ingredient in our AIξ model is sequential decision theory. We do not need much more than the maximum expected utility principle and the expectimax algorithm [Mic66, RN95]. The book of von Neumann and Morgenstern [NM44] might be seen as the initiation of game theory, which already contains the expectimax algorithm as a special case. The literature on reinforcement learning and sequential decision theory is vast and we refer to the references given in the textbooks [SB98, BT96].

The author's contributions. Most of the issues addressed in this article can already be found scattered in various reports and publications by the author: The AIξ model has first been introduced and discussed in March 2000 in [Hut00] in a 62 page long report. More succinct descriptions have been published in [Hut01d, Hut01e]. The AIξ model has been argued to formally solve a number of problem classes, including sequence prediction, strategic games, function minimization, reinforcement and supervised learning [Hut00]. A variant of AIξ has recently been shown to be self-optimizing and Pareto-optimal [Hut02c]. The construction of a general fastest algorithm for all well-defined problems [Hut02a] arose from the construction of the time-bounded AIξ^{tl} model [Hut01d]. Tight [Hut02b] error [Hut01c, Hut01a] and loss [Hut01b] bounds for Solomonoff's universal sequence prediction scheme have been proven. Loosely related ideas on a market/economy based reinforcement learner [KHS01b] and gradient based reinforcement planner [KHS01a] have been implemented. These and other papers are available at http://www.idsia.ch/~marcus/ai.

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